COMMUTATIVE COCYCLES AND STABLE BUNDLES OVER SURFACES

DANIEL A. RAMRAS AND BERNARDO VILLARREAL

Abstract. Commutative K–theory, a cohomology theory built from spaces of commuting matrices, has been explored in recent work of Adem, Gómez, Gritschacher, Lind, and Tillman. In this article, we use unstable methods to construct explicit representatives for the real commutative K–theory classes on surfaces. These classes arise from commutative $O(2)$–valued cocycles, and are analyzed via the point-wise inversion operation on commutative cocycles.

Contents

2010 *Mathematics Subject Classification.* Primary 55N15; Secondary 55R35.
Key words and phrases. Commutative K–theory, vector bundle, commuting matrices.
D. Ramras was partially supported by the Simons Foundation (Collaboration Grant #279007).

This is the author's manuscript of the article published in final edited form as:
Ramras, D. A., & Villarreal, B. (2019). Commutative cocycles and stable bundles over surfaces. Forum Mathematicum, 31(6), 1395–1415. https://doi.org/10.1515/forum-2018-0163

1. Introduction

Let X be a finite CW complex. In Adem–Gómez [3], commuting orthogonal matrices were used to construct a variant of real topological K–theory, $KO_{\mathrm{com}}(X)$, whose classes are represented by vector bundles over X equipped with a *commutative trivialization.* A trivialization of a bundle $E \to X$ is called *commutative* if all pairs of transition functions commute (as elements of the structure group) wherever they are simultaneously defined (see Definition 2.1). Similar considerations with unitary matrices yield complex commutative K–theory. There is a natural forgetful map $KO_{\mathrm{com}}(X) \to KO^0(X)$, and in Adem–Gómez–Lind–Tillman [4] it was shown that this map admits a splitting (additively), and the same holds in the complex case.

Using methods from stable homotopy theory, Gritschacher [9] showed that the reduced group $\widetilde{KO}_{\mathrm{com}}(S^2)$ is isomorphic to $\mathbb{Z}/2 \oplus \widetilde{KO}(S^2)$. One of our main results, Theorem 4.9, is that this holds with S^2 replaced by any closed, connected surface Σ, and in fact our unstable methods provide a new proof of Gritschacher's result. Moreover, in Theorem 5.5 we establish an isomorphism of non-unital rings

$$\widetilde{KO}_{\mathrm{com}}(\Sigma) \cong \widetilde{KO}(\Sigma) \times \langle y \rangle \tag{1}$$

with $y^2 = 2y = 0$, and in the Appendix we give a presentation for the real K–theory ring of Σ.

We use *commutative cocycles* with values in $O(2)$ to give an explicit construction of the generator

$$y \in \ker(KO_{\mathrm{com}}(\Sigma) \longrightarrow KO(\Sigma)).$$

The crucial feature of a commutative cocycle is that its point-wise inverse is again a well-defined, commutative cocycle. To construct y, we exhibit an explicit open cover of S^2 along with a commutative cocycle α whose associated vector bundle is *trivial*, but whose point-wise inverse produces a *non*-trivial bundle (Proposition 3.4). The cocycle α is classified by a map $S^2 \to B_{\mathrm{com}}O(2)$, where $B_{\mathrm{com}}O(2) \subset BO(2)$ is the *classifying space for commutativity* in $O(2)$ introduced by Adem–Cohen–Torres-Giese [2] (see Section 2). By analyzing the inclusions

$$O(2) \hookrightarrow O(3) \hookrightarrow \cdots \hookrightarrow O$$

and the associated maps on $B_{\mathrm{com}}O(n)$ (Proposition 4.4), we show that α corresponds to a non-trivial class $y \in \widetilde{KO}_{\mathrm{com}}(S^2)$; since the vector bundle associated to α is trivial, the image of y in $\widetilde{KO}(S^2)$ is trivial.

Recent work of Antolín-Camarena, Gritschacher, and Villarreal [5] introduces new characteristic classes for bundles equipped with commutative trivializations, and we use these classes to extend our result from S^2 to general surfaces. In the last section, we apply our methods to deduce new information about these characteristic classes, which are then used to deduce the ring isomorphism (1).

Remark 1.1. In the complex case, the natural map $\widetilde{KU}_{\mathrm{com}}(X) \to \widetilde{KU}(X)$ is an isomorphism for every 2-dimensional CW complex X. This follows from Adem–Gómez [3, Proposition 3.3].

Acknowledgments. We thank Simon Gritschacher for helpful conversations.

2. Transitionally commutative structures

Let G be a Lie group. For any integer $k > 0$, let $C_k(G) \subset G^k$ be the subspace of ordered commuting k-tuples in G. Defining $C_0(G) := \{e\}$, the family of spaces $\{C_k(G)\}_{k\geq 0}$ has a simplicial structure arising by restriction of the simplicial structure of NG via the inclusions $C_k(G) \hookrightarrow (NG)_k$, where NG is the nerve of G as a category with one object. The *classifying space for commutativity* of G is defined as the geometric realization

$$B_{\mathrm{com}}G := |C_\bullet(G)|.$$

For a finite CW complex X, the *reduced real commutative* K–theory of X is defined as

$$\widetilde{KO}_{\mathrm{com}}(X) = \mathrm{colim}_{n\to\infty}[X, B_{\mathrm{com}}O(n)].$$

The reduced complex commutative K–theory of X is defined similarly, replacing $O(n)$ by $U(n)$.

In general, the space $B_{\mathrm{com}}G$ comes naturally equipped with an inclusion

$$\iota\colon\ B_{\mathrm{com}}G \to BG.$$

The pullback of the universal G-bundle $EG \to BG$ along ι is denoted $E_{\mathrm{com}}G$. Since EG is contractible, we have that

$$E_{\mathrm{com}}G = \mathrm{hofib}(B_{\mathrm{com}}G \to BG).$$

In the Appendix, we show that the kernel of $\iota_*\colon\ \widetilde{KO}_{\mathrm{com}}(X) \to \widetilde{KO}(X)$ is naturally isomorphic to $[X, E_{\mathrm{com}}O]$, so that $E_{\mathrm{com}}O$ represents the "non-standard" part of real commutative K–theory (and similarly in the complex case).

Definition 2.1. Let $E \to X$ be a principal G-bundle. A *commutative trivialization* of E is an open cover $\{U_i\}$ of X together with transition functions

$$\{\alpha_{ij}\colon\ U_i \cap U_j \to G\}$$

such that

(1) E is isomorphic to the G-bundle induced by $\{\alpha_{ij}\}$;
(2) For any $x \in U_i \cap U_j \cap U_k \cap U_l$, the commutator $[\alpha_{ij}(x), \alpha_{kl}(x)]$ is trivial.

In general, we say say that a cocycle $\alpha = \alpha_{ij}$ is commutative if condition (2) above holds.

Following [3] a principal G-bundle that admits a commutative trivialization is called *transitionally commutative.* In [9], S. Gritschacher defines a *transitionally commutative structure* (TC structure) on a bundle $E \to X$, as a homotopy class $[\widetilde{f}] \in [X, B_{\mathrm{com}}G]$ such that $\iota \circ \widetilde{f}$ is a classifying map for E; in other words, a TC structure on E is a lift, up to homotopy, of a classifying map f for E:

$$\begin{tikzcd} & B_{\mathrm{com}}G \arrow[d, "\iota"] \\ X \arrow[ur, bend left, "f'"] \arrow[r, "f"] & BG. \end{tikzcd}$$

Definition 2.2. For every abelian subgroup $A \subset G$, the inclusion $BA \to BG$ factors as

$$BA = B_{\mathrm{com}}A \to B_{\mathrm{com}}G \xrightarrow{\iota} BG.$$

If a classifying map $f\colon X \to BG$ for a G-bundle E factors up to homotopy through the inclusion $BA \to BG$, then we say the lift $f'\colon X \to BA \to B_{\mathrm{com}}G$ is an *algebraic* TC structure on E.

Example 2.3. Three particular examples of algebraic TC structures will be important. If E is a trivial bundle, then it has a *standard* algebraic TC structure in which $A = \{e\}$ is the trivial subgroup of G. If E is a real line bundle, then its classifying map factors uniquely (up to homotopy) through $BO(1)$, again yielding a *standard* algebraic TC structure on E. Finally, if E is a 2-dimensional real vector bundle, then a choice of orientation on E yields an algebraic TC structure with $A = SO(2)$, and again we refer to this as the standard TC structure associated to an oriented 2–dimensional bundle. In these situations, we will use the notation E^{st} to denote E with its standard TC structure.

The following theorem allows us to pass between the homotopical notion of a TC structure and the more geometric notion in Definition 2.1.

Theorem 2.4 ([3], Adem, Gómez)**.** *Let X be a finite CW-complex and $E \to X$ a principal G-bundle with classifying map $f\colon X \to BG$. Then f factors up to homotopy through the inclusion $B_{\mathrm{com}}G \to BG$ if and only if E is transitionally commutative.*

Given a commutative trivialization α with respect to a cover $\mathcal{U}$ on a bundle E, we will denote by f_α the transitionally commutative structure defined by

$$f_\alpha := |N(\alpha)| \circ \lambda,$$

where $\lambda\colon\ X \to N(\mathcal{U})$ is the homotopy equivalence associated to a partition of unity $\{\lambda\}$ on X subordinate to $\mathcal{U}$. This is independent of the choice of partition of unity $\{\lambda\}$ (up to homotopy) since it can easily be verified that given another partition of unity $\{\lambda'\}$ subordinate to $\mathcal{U}$, the induced maps $\lambda, \lambda'\colon X \to |N(\mathcal{U})|$ are homotopic (via a linear homotopy). We will say that two commutative trivializations are equivalent if the induced transitionally commutative structures are homotopic.

2.1. **Power operations.** The simplicial space $C_\bullet(G)$ has a nice feature with respect to power maps $(-)^n\colon\ G \to G$ (when $n = 0$, this is the constant map $g \mapsto e$). For each n we have a simplicial map

$$(\phi^n)_\bullet\colon\ C_\bullet(G) \to C_\bullet(G),$$

given by $(\phi^n)_k(g_1, ..., g_k) = (g_1^n, ..., g_k^n)$. After taking geometric realization we then see a map

$$\phi^n\colon\ B_{\mathrm{com}}G \to B_{\mathrm{com}}G.$$

The following properties can easily be checked.

(1) $\phi^m \circ \phi^n = \phi^{mn}$
(2) Any map $\gamma\colon\ B_{\mathrm{com}}G \to B_{\mathrm{com}}H$ arising from a homomorphism of Lie groups $G \to H$ satisfies $\phi^n \circ \gamma = \gamma \circ \phi^n$.

Lemma 2.5. *Suppose E is a G-bundle with a commutative trivialization whose transition functions are $\{\alpha_{ij}\}$. Then for any integer n, the set $\alpha^n := \{(-)^n \circ \alpha_{ij}\}$ is a commutative cocycle with associated transitionally commutative structure $[f_{\alpha^n}] = [\phi^n \circ f_\alpha]$.*

Proof. Suppose α is a commutative cocycle with respect to a cover $\mathcal{U}$. Then the induced simplicial map satisfies $N(\alpha)\colon N(\mathcal{U}) \to C_\bullet(G) \subset NG$. The simplicial map induced by α^n factors through

$$N(\mathcal{U}) \xrightarrow{N(\alpha)} C_\bullet(G) \xrightarrow{\phi^n_\bullet} C_\bullet(G) \subset NG$$

Therefore $|N(\alpha^n)| = \phi^n \circ |N(\alpha)|$. $\square$

3. Transitionally commutative structures over the sphere

In this section we exhibit a commutative trivialization for the trivial $O(2)$–bundle over S^2, with the property that the induced map $S^2 \to B_{\mathrm{com}}O(2)$ is *not* nullhomotopic. We prove this last statement using the power operations introduced in the previous section. In Section 4 this TC structure will be used to produce non-trivial commutative K–theory classes on surfaces.

3.1. **Associated clutching function over a sphere.** We use the cover $\mathcal{C}$ of $S^n \subset \mathbb{R}^{n+1}$ with $n > 1$, consisting of three closed sets C_1, C_2 and C_3 defined as follows. Consider the "left" and "right" hemispheres

$$C_1 = \{\vec{x} \in S^n \mid x_0 \leq 0\} \quad \text{and} \quad C = \{\vec{x} \in S^n \mid x_0 \geq 0\}.$$

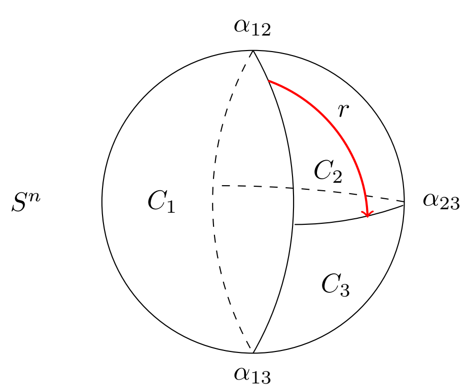
Next, we cover C with the "north" and "south" n-dimensional discs,

$$C_2 = \{\vec{x} \in C \mid x_n \geq 0\} \text{ and } C_3 = \{\vec{x} \in U \mid x_n \leq 0\}.$$

Consider the retraction $r\colon C_2 \to C_2 \cap C_3$ given by

$$r(x_0, ..., x_n) = \left(\sqrt{1 - \sum_{i=1}^{n-1} x_i^2}, x_1, \dots, x_{n-1}, 0\right)$$

Let $\alpha = \{\alpha_{ij}\colon C_i \cap C_j \to G\}$ be a G–valued cocycle on the cover $\mathcal{C}$. (Note that we are using a covering by closed sets for convenience, and the notion of cocycle does not require openness.)



We define the clutching function $\varphi\colon C_1 \cap (C_2 \cup C_3) = S^{n-1} \to G$ as

$$\varphi(x) = \begin{cases} \alpha_{12}(x)\alpha_{23}(r(x)) & \text{if} \quad x \in C_1 \cap C_2 \\ \alpha_{13}(x) & \text{if} \quad x \in C_1 \cap C_3 \end{cases}$$

which is well defined and continuous since α satisfies the cocycle condition. We will refer to φ as the clutching function induced by α.

Lemma 3.1. *The principal G-bundle over S^n induced by α is isomorphic to the bundle clutched by $\varphi\colon S^{n-1} \to G$.*

Proof. Consider the G-bundle induced by α,

$$E = C_1 \times G \sqcup C_2 \times G \sqcup C_3 \times G/(x,g)_j \sim (x, \alpha_{ij}(x)g)_i$$

where $(x,g)_i \in C_i \times G$. Denote an element in $(C_2 \cup C_3) \times G$ by $(x,g)_{23}$, and let

$$\psi\colon\ C_1 \times G \sqcup (C_2 \cup C_3) \times G \to E$$

be the map given by $\psi((x,g)_1) = [(x,g)_1]$ and

$$\psi((x,g)_{23}) = \begin{cases} [(x, \alpha_{23}(r(x))g)_2] & \text{if} \quad x \in C_2 \\ \quad [(x,g)_3] & \text{if} \quad x \in C_3 \end{cases}$$

The map ψ is well defined in the quotient

$$E_\varphi = (C_1 \times G \sqcup (C_2 \cup C_3) \times G) / (x,g)_{23} \sim (x, \varphi(x)g)_1.$$

Indeed, let $x \in C_1 \cap (C_2 \cup C_3)$. Then $\psi((x,g)_1) = [(x,g)_1]$ and we have two cases:

- if $x \in C_2$, $\psi((x, \varphi(x)^{-1}g)_{23}) = [(x, \alpha_{12}(x)^{-1}g)_2] = [(x,g)_1]$
- if $x \in C_3$, $\psi((x, \varphi(x)^{-1}g)_{23}) = [(x, \alpha_{13}(x)^{-1}g)_3] = [(x,g)_1]$

Let ψ also denote the induced map $E_\varphi \to E$. To prove our claim it only remains to show that ψ is right G-equivariant, but this is true since the construction of ψ only involves left multiplication by transition functions. $\square$

3.2. **TC structures on $O(2)$-bundles over S^2.** We study the example case of $\mathcal{C}$ as a cover of S^2. In this case the pairwise intersections of the three sets in $\mathcal{C}$ are all homeomorphic to a closed interval. Let k be an integer and denote the matrix $\begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix}$ by $R_{k\theta}$, where $0 \leq \theta \leq \pi$. Also let $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. With these matrices we can define transition functions over $\mathcal{C}$ as follows. First we parametrize each of the intersections $C_1 \cap C_2$ and $C_1 \cap C_3$ by $\theta \in [0, \pi]$, in such a way that the front side intersection point $(0,1,0)$ corresponds to $\theta = 0$ and the back side intersection point $(0,-1,0)$ corresponds to $\theta = \pi$. We can then define

$$\begin{aligned} {}_k\alpha_{12}(\theta) &= R_{k\theta} \\ {}_k\alpha_{23}(x) &= A \\ {}_k\alpha_{13}(\theta) &= R_{k\theta}A \end{aligned} \tag{2}$$

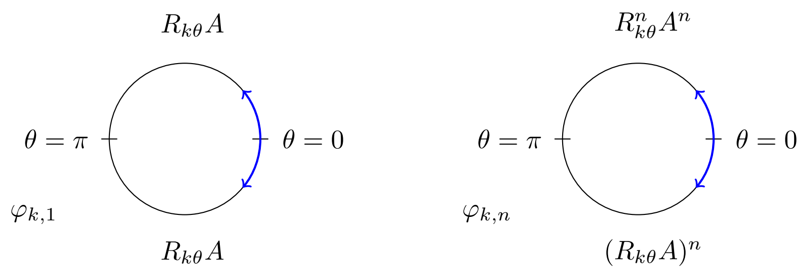
Lemma 3.2. *The set of transition functions $\{{}_k\alpha_{ij}\}$ is a commutative cocycle for the trivial $O(2)$-bundle over S^2.*

Remark 3.3. Below, we will need to replace the closed cover $\mathcal{C}$ on S^2 by an open cover. We can do this by taking neighborhoods of the sets C_i: specifically, choose $\epsilon > 0$ and let $U_1 = \{\vec{x} \in S^n \,|\, x_0 < \epsilon\}$, $U_2 = \{\vec{x} \in S^n \,|\, x_2 > -\epsilon\}$, and $U_3 = \{\vec{x} \in S^n \,|\, x_2 < \epsilon\}$. These sets deformation retract back to the closed sets C_i, and on their triple intersection these deformation retractions can be arranged to have image inside an ϵ–neighborhood of $\{(0,1,0),(0,-1,0)\}$ (in the l_1–metric on $\mathbb{R}^3$). We can now reparametrize the function R_θ so that it is constant in a 2ϵ–neighborhood of $\{(0,1,0),(0,-1,0)\}$. Then by precomposing with these deformation retractions, we obtain an extension of $\{{}_k\alpha_{ij}\}$ to an open cover. This extension has the same associated vector bundle (up to canonical isomorphism).

Before proving Lemma 3.2, we set up some notation. For each $k, n \in \mathbb{Z}$, let

$$\varphi_{k,n} \colon S^1 \to O(2)$$

denote the clutching function induced by ${}_k\alpha^n$, as in Section 3.1. We can represent $\varphi_{k,n}$ diagrammatically as



(3)

where the upper semicircle corresponds to $U_1 \cap U_2$ and the lower semicircle to $U_1 \cap U_3$.

Proof of Lemma 3.2. Notice that $R_0 = I$ and $R_{k\pi} = \pm I$, that is, at the intersection points $x \in U_1 \cap U_2 \cap U_3$, both ${}_k\alpha_{12}(x)$ and ${}_k\alpha_{13}(x)$ lie in the center of $O(2)$. Therefore ${}_k\alpha$ is a commutative cocycle.

Running around the circle on the right of Figure (3) in counterclockwise orientation starting at $\theta = 0$, we see that the loop determined by $\varphi_{k,1}$ is homotopic to the constant loop for all $k \in \mathbb{Z}$. The result now follows from Lemma 3.2. $\square$

Notation: From now on we will simply denote $f_k := f_{{}_k\alpha} \colon S^2 \to B_{\mathrm{com}}O(2)$. By Lemma 2.5, we have that $[\phi^n \circ f_k] = [f_{({}_k\alpha)^n}]$.

The definition of R_θ is made in such a way that the homotopy class $[R_{2\theta}]$ can be regarded as a generator in $\pi_1(SO(2))$ and thus a generator in $\pi_1(O(2))$ via the inclusion $SO(2) \subset O(2)$. We fix a generator $1 \in \pi_2(BO(2)) = \mathbb{Z}$ by choosing it to be the pre-image of the aforementioned generator under the inverse of the connecting homomorphism $\pi_2(BO(2)) \to \pi_1(O(2))$ arising from the long exact sequence of homotopy groups of the fibration $O(2) \to EO(2) \to BO(2)$.

Proposition 3.4. *The homotopy class of the classifying map*

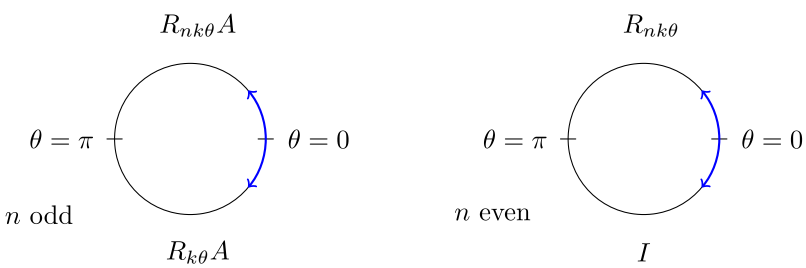
$$S^2 \xrightarrow{\phi^n \circ f_k} B_{\mathrm{com}}O(2) \xrightarrow{\iota} BO(2)$$

in $\pi_2(BO(2))$ is given by $nk/2$ if n is even and $(n-1)k/2$ if n is odd.

Proof. By Lemma 3.1, the bundle classified by $\iota \circ \phi^n \circ f_k$ is isomorphic to the bundle clutched by $\varphi_{k,n}$ as defined above. Notice that $AR_{k\theta}A = R_{-k\theta}$ and $A^2 = I$, so that

$$(R_{k\theta}A)^n = \begin{cases} I & n \text{ even} \\ R_{k\theta}A & n \text{ odd}, n > 0 \\ AR_{-k\theta} & n \text{ odd}, n < 0 \end{cases} \qquad \text{and} \qquad R^n_{k\theta}A^n = \begin{cases} R_{nk\theta} & n \text{ even} \\ R_{nk\theta}A & n \text{ odd} \end{cases}$$

Since $R_{k\theta}A = R_{-k\theta}A$, then for all n odd $(R_{k\theta}A)^n = R_{k\theta}A$. The loops $\varphi_{k,n}$ in $O(2)$ now have the form



(4)

When n is even, $\varphi_{k,n}$ actually lies in $SO(2)$, and thus the homotopy class of the classifying map of the bundle clutched by $\varphi_{k,n}$ is given by $\deg(R_{nk\theta}) = nk/2$.

To analyze odd powers, first, notice that in counterclockwise orientation, the loop $\phi_{k,n}$ is the concatenation $\overline{R_{k\theta}A} * R_{nk\theta}A = R_{k(n-1)\theta}A$ (here we use $\overline{\gamma}$ to denote the reverse of a path γ).When n is odd, we claim that the homotopy class of the classifying map of the bundle clutched by $\varphi_{k,n}$ is given by $\deg(R_{k(n-1)\theta})$. To show this, we prove that the $O(2)$-bundle clutched by $R_{k(n-1)\theta}A$ is isomorphic to the $O(2)$-bundle clutched by $R_{k(n-1)\theta}$. Consider the map

$$L\colon U_1 \times O(2) \sqcup (U_2 \cup U_3) \times O(2) \to U_1 \times O(2) \sqcup (U_2 \cup U_3) \times O(2)$$

given by $L((x,g)_1) = (x,g)_1$ and $L((x,g)_{23}) = (x,Ag)_{23}$, where the subscripts indicate the factor of the disjoint union containing the element. The bundle induced by $R_{k(n-1)\theta}A$ is clutched by the relation

$$(x,g)_{23} \sim (x, R_{k(n-1)\theta}Ag)_1.$$

Now in the bundle clutched by $R_{n(k-1)\theta}$ we have that

$$L((x,g)_{23}) = (x,Ag)_{23} \sim (x, R_{k(n-1)\theta}(Ag))_1 = L((x, R_{k(n-1)\theta}Ag)_1)$$

and thus L is well defined in the quotient. Since L is given by left multiplication only, it is clearly right equivariant, giving the desired isomorphism. Therefore

$$[\iota \circ \phi^n \circ f_k] = \begin{cases} \dfrac{nk}{2} & n \text{ and even} \\ \dfrac{(n-1)k}{2} & n \text{ odd.} \end{cases}$$

□

Now we describe the behaviour of the power operations ϕ^n for $G = SO(2)$ in homotopy groups, which will be useful for our calculations.

Lemma 3.5. *For every integer n, there is a commutative diagram*

$$\begin{tikzcd} \pi_2(BSO(2)) \arrow[r, "\delta", "\cong"'] \arrow[d, "\phi^n_*"'] & \pi_1(SO(2)) \arrow[d, "(-)^n_*"] \\ \pi_2(BSO(2)) \arrow[r, "\delta", "\cong"'] & \pi_1(SO(2)) \end{tikzcd}$$

where the horizontal maps δ are both the connecting homomorphism in the long exact sequence of homotopy groups associated to the fibration $SO(2) \to ESO(2) \to BSO(2)$. Consequently $(\phi^n)^*\colon H^2(BSO(2);\mathbb{Z}) \to H^2(BSO(2);\mathbb{Z})$ is multiplication by n.

Proof. A homomorphism of Lie groups $f\colon G \to H$ induces maps $E(f)\colon EG \to EH$ and $B(f)\colon BG \to BH$ that fit into a commutative diagram of fibrations, and the restriction of $E(f)$ to the fibers agrees with f. Since $SO(2)$ is abelian, the power maps $(-)^n\colon SO(2) \to SO(2)$ are homomorphisms and by definition $\phi^n = B((-)^n)$. Then by naturality, $(-)^n_*$ and ϕ^n_* fit into a morphism of long exact sequences in homotopy groups making the above diagram commutative. Since $BSO(2)$ is simply connected, the last statement in the lemma follows from the Hurewicz Theorem and the Universal Coefficient Theorem. □

Any $SO(2)$-bundle has a standard algebraic TC structure given by the composition of its classifying map and the inclusion $j\colon\ BSO(2) \to B_{\mathrm{com}}O(2)$ induced by the inclusion $SO(2) \hookrightarrow O(2)$. Since an $O(2)$-bundle over S^2 is orientable, it has an algebraic TC structure given by a choice of orientation $m = [g_m] \in \pi_2(BSO(2))$, that is, a lift $g_m\colon\ S^2 \to BSO(2)$ of the classifying map composed with j.

Now we can compute the values of the maps $f_k\colon S^2 \to B_{\mathrm{com}}O(2)$ in integral cohomology. This will allow us to construct non-trivial TC structures on bundles over other closed, connected surfaces.

Lemma 3.6. *For every k, $j \circ g_k \simeq \phi^2 \circ f_k$.*

Proof. Recall that f_k is the map induced by the set of transitionally commutative functions ${}_k\alpha_{ij}$ in (2). The effect of post-composing with ϕ^2 can be seen at the level of simplicial spaces. Since ${}_k\alpha_{12}^2(\theta) = (R_{k\theta})^2 = (R_{2k\theta})$, ${}_k\alpha_{23}^2(x) = A^2 = I$ and ${}_k\alpha_{13}^2(\theta) = (R_{k\theta}A)^2 = I$ we can conclude that $\phi^2_\bullet \circ N({}_k\alpha_{ij})\colon\ N(\mathcal{U}) \to N(O(2))$ has image contained in $N(SO(2))$. Then the homotopy class of $\phi^2 \circ f_k$ is determined by the degree of the map in $\pi_2(BSO(2))$ which by Proposition 3.4 is the same as the degree of $j \circ g_k$. □

In [5], the authors show that $\iota^*\colon\ H^*(BO(2);\mathbb{Z}) \to H^*(B_{\mathrm{com}}O(2);\mathbb{Z})$ is injective, and establish the existence of a cohomology class $r \in H^2(B_{\mathrm{com}}O(2);\mathbb{Z})$ satisfying

$$j^*(r) = 2e \tag{5}$$

where $e \in H^2(BSO(2);\mathbb{Z})$ is the Euler class of the universal oriented $SO(2)$-bundle, and $j\colon BSO(2) \to B_{\mathrm{com}}O(2)$ is the inclusion as before.

Lemma 3.7. *Let E_k denote the oriented rank 2 vector bundle of degree k over S^2, and let $e(E_k)$ denote its Euler class. Then*

$$f_k^*(r) = e(E_k).$$

Proof. By Lemma 3.6, in integral cohomology $(\phi^2 \circ f_k)^*$ has the same values as

$$H^2(B_{\mathrm{com}}O(2);\mathbb{Z}) \xrightarrow{j^*} H^2(BSO(2);\mathbb{Z}) \xrightarrow{g_k^*} H^2(S^2;\mathbb{Z})$$

which by (5) maps $r \to 2e(E_k)$, where $e(E_k)$ is the Euler class of the bundle over S^n induced by g_k. By Lemma 3.5, $(\phi^2)^*(r) = 2r$, and then under $(\phi^2 \circ f_k)^*$, $r \mapsto 2f_k^*(r)$, so we can conclude that $f_k^*(r) = e(E_k)$. □

Example 3.8. It is easy to check that given a commutative G–valued cocycle $\{\alpha_{ij}\}$ on a space Y, with associated G-bundle E, and a continuous map $f\colon X \to Y$, then $\{\alpha_{ij} \circ f\}$ defines a commutative cocycle on X with associated bundle $f^*(E)$. Let Σ be an orientable, closed, connected surface and $c\colon \Sigma \to S^2$ the map that collapses the complement of an open disk in Σ to a point. Since Σ is orientable, c is a degree 1 map. Lemma 3.7 shows that for every integer k, we have

$$c^* f_k^*(r) = c^*(e(E_k)) = kc^*(e(E_1)).$$

Moreover, $e(E_1) \in H^2(S^2;\mathbb{Z})$ is a generator (by definition), and hence so is $c^*(e(E_1)) \in H^2(\Sigma;\mathbb{Z})$. Thus the TC structures $f_k \circ c$ all lie in different homotopy classes. In other words, for every k the commutative cocycles $\{{}_k\alpha_{ij} \circ c\}$ on the trivial $O(2)$-bundle over Σ define different TC structures.

Similarly, if Σ is a a closed, connected, non-orientable surface, we can choose a map $c\colon \Sigma \to S^2$ as above, and c will have degree 1 in cohomology with coefficients in $\mathbb{Z}/2$, and we find that for k odd, $c^* f_k^*(\bar{r})$ generates $H^2(\Sigma;\mathbb{F}_2)$, where $\bar{r}$ is the image of r in $\mathbb{F}_2$–cohomology. In particular, we find that the TC structures $\{{}_k\alpha_{ij} \circ c\}$ are non-trivial when k is odd.

In [5] it is shown that $\pi_2(B_{\text{com}}O(2)) \cong \mathbb{Z}^3$. In addition to the TC structures $f_k\colon S^2 \to B_{\text{com}}O(2)$ on the trivial bundle, we also have the standard algebraic TC structures $j \circ g_m$ considered in Lemma 3.6. It is an interesting question whether combinations of these TC structures yield all homotopy classes in $\pi_2(B_{\text{com}}O(2))$. To be more precise about these combinations, we have

$$S^2 \xrightarrow{\text{pinch}} S^2 \vee S^2 \xrightarrow{f_k \vee g_m} B_{\text{com}}O(2) \vee BSO(2) \xrightarrow{Id,j} B_{\text{com}}O(2)$$

which is nothing but the sum in $\pi_2(B_{\text{com}}O(2))$. Motivated by this, we will denote the above TC structure by $f_k + (j \circ g_m)$. Here we prove that each pair of integers (k, m) yields a distinct class $[f_k] + [(j \circ g_m)] \in \pi_2(B_{\text{com}}O(2))$.

Proposition 3.9. *Let E_m be an $O(2)$-bundle over S^2 with a choice of orientation given by $m \in \pi_2(BSO(2))$. Then for any $k \geq 0$, $f_k + (j \circ g_m)$ is a TC structure on E_m. Moreover, for every k these are all different TC structures on E_m.*

Proof. Let $h_m\colon S^2 \to BO(2)$ be a classifying map of E_m. Then $\iota \circ (j \circ g_m) \simeq h_m$ and by Lemma 3.2, $\iota \circ f_k$ is homotopic to the constant map so that indeed, $f_k + (j \circ g_m)$ is a lift of h_m to $B_{\text{com}}O(2)$. To see that all these lifts are in different homotopy classes in $\pi_2(B_{\text{com}}O(2))$, consider the compositions $(\iota \circ \phi^{-1}) \circ (f_k + (j \circ g_m))$. Notice that the inclusion j commutes with the map ϕ^{-1} since j is induced from the homomorphism $SO(2) \hookrightarrow O(2)$. Therefore $\phi^{-1} \circ j \circ g_k = j \circ \phi^{-1} \circ g_k$. By Lemma 3.5, $\phi_*^{-1}[g_m] = -[g_m]$. Then by Proposition 3.4 we see that in $\pi_2(BO(2))$ we have

$$[(\iota \circ \phi^{-1}) \circ (f_k + (j \circ g_m))] = -k - m,$$

which are distinct for different values of k. $\square$

4. Real commutative K-theory of surfaces: Additive structure

In this section we prove one of our main results, computing the group $\widetilde{KO}_{\text{com}}(\Sigma)$ for closed connected surfaces. The key ingredient is to compute the homotopy groups $\pi_2(B_{\text{com}}O(n))$ for $n \geq 3$ together with the homomorphisms $\pi_2(B_{\text{com}}O(2)) \to \pi_2(B_{\text{com}}O(n))$ induced by the canonical inclusions $O(2) \hookrightarrow O(n)$.

4.1. **Stability.** We study how the classes f_k considered in the previous section behave as we stabilize using the inclusions $O(2) \hookrightarrow O(n)$ given by $X \mapsto \begin{pmatrix} X & 0 \\ 0 & I_{n-2} \end{pmatrix}$. These maps induce maps $i_n\colon BO(2) \to BO(n)$ and $j_n\colon B_{\text{com}}O(2) \to B_{\text{com}}O(n)$.

Proposition 4.1. *If k is odd and $n \geq 3$, the homotopy class of*

$$j_n \circ f_k \colon\ S^2 \to B_{\mathrm{com}}O(n)$$

is non-trivial in $\pi_2(B_{\mathrm{com}}O(n))$.

Proof. Fix $n \geq 3$. We have a commutative diagram

$$\begin{tikzcd} B_{\mathrm{com}}O(2) \arrow[r, "j_n"] \arrow[d, "\iota_2"] & B_{\mathrm{com}}O(n) \arrow[d, "\iota_n"] \\ BO(2) \arrow[r, "i_n"] & BO(n) \end{tikzcd}$$

in which ι_2 and ι_n are the natural inclusions, and j_n is the map on $B_{\mathrm{com}}(-)$ induced by i_n. By Lemma 3.2, $[\iota_2 \circ f_k]$ is a trivial class in $\pi_2(BO(2))$, so that $(i_n)_*[\iota_2 \circ f_k]$ is also trivial in $\pi_2(BO(n))$, which is the same class as $(\iota_n)_*[j_n \circ f_k]$.

To study the homotopy class of $j_n \circ f_k \colon\ S^2 \to B_{\mathrm{com}}O(n)$ in $\pi_2(B_{\mathrm{com}}O(n))$, let η be the composition

$$S^2 \xrightarrow{j_n \circ f_k} B_{\mathrm{com}}O(n) \xrightarrow{\phi^{-1}} B_{\mathrm{com}}O(n) \xrightarrow{\iota_n} BO(n).$$

We claim that the homotopy class of η is the nontrivial element in $\pi_2(BO(n)) = \mathbb{Z}/2$ if and only if k is odd. Notice that this would finish the proof, since our claim forces $[j_n \circ f_k]$ to be non-trivial as long as k is odd.

The map j_n is induced by a homomorphism, so by naturality it commutes with ϕ^{-1}. Now we see that

$$\eta = \iota_n \circ \phi^{-1} \circ j_n \circ f_k = \iota_n \circ j_n \circ \phi^{-1} \circ f_k = i_n \circ \iota_2 \circ \phi^{-1} \circ f_k.$$

Proposition 3.4 implies that the homotopy class of $\iota_2 \circ \phi^{-1} \circ f_k$ in $\pi_2(BO(2))$ is represented by the integer $-k$. Thus to prove our claim we only need to recall how i_n behaves on $\pi_2(-)$. The long exact sequence of homotopy groups of the fibrations $S^m \to BO(m) \to BO(m+1)$ imply that $(i_3)_* \colon \pi_2(BO(2)) \to \pi_2(BO(3))$ is the reduction mod 2 homomorphism, and for $m > 2$, $\pi_2(BO(m)) \to \pi_2(BO(m+1))$ is an isomorphism. It follows that $(i_n)_* \colon \pi_2(BO(2)) \to \pi_2(BO(n))$ is reduction mod 2 as well. Hence $[\eta] = (i_n)_*[\iota_2 \circ \phi^{-1} \circ f_k] \in \pi_2(BO(n))$ is a generator if and only if k is odd. $\square$

4.2. **Computing $\pi_2(B_{\mathrm{com}}SO(n))$ and $\pi_2(B_{\mathrm{com}}O(n))$.** Recall that in $\mathcal{C}_\bullet(G)$, the simplicial model for $B_{\mathrm{com}}G$, the face maps $d_i \colon \mathcal{C}_n(G) \to \mathcal{C}_{n-1}(G)$ are given by

$$d_i(g_0, \ldots, g_n) = \begin{cases} (g_1, \ldots, g_n) & i = 0 \\ (g_0, \ldots, g_{i-1}g_i, \ldots, g_n) & 0 < i < n \\ (g_0, \ldots, g_{n-1}) & i = n. \end{cases}$$

These maps induce a simplicial abelian group structure on $H_q(\mathcal{C}_\bullet(G))$ for any $q \geq 0$.

Lemma 4.2. *Let G be a connected Lie group. Then*

$$H_2(B_{\mathrm{com}}G; \mathbb{Z}) = H_2 H_0(\mathcal{C}_\bullet(G)) \oplus H_1(G; \mathbb{Z}).$$

Proof. Let us write $H_m(X)$ for singular homology with integral coefficients. We use the homology spectral sequence $E^r_{p,q}$ associated to the simplicial space $\mathcal{C}_\bullet(G)$ that strongly converges to $H_{p+q}(|\mathcal{C}_\bullet(G)|) = H_{p+q}(B_{\mathrm{com}}G)$. Since $\mathcal{C}_\bullet(G)$ is proper (see [3, Appendix]), the E^2-page is given by $E^2_{p,q} = H_p(H_q(\mathcal{C}_\bullet(G)))$ where for each q, $H_q(\mathcal{C}_\bullet(G))$ is regarded as a simplicial abelian group. Let us write $A_n = H_q(\mathcal{C}_n(G))$.

Then $E^2_{p,q} = H_p((A_*, \partial))$ is the p-th homology group of the (un-normalized) chain complex (A_*, ∂) with differentials given by $\partial = \sum_{i=0}^n (-1)^i (d_i)_*$. To compute $H_2(B_{\mathrm{com}}G)$ we only need to worry about $E^2_{0,2}, E^2_{1,1}$ and $E^2_{2,0}$. We study each of these cases separately.

- $E^2_{0,2} = H_0H_2(\mathcal{C}_\bullet(G))$. In this case $A_0 = H_2(\mathcal{C}_0(G)) = H_2(pt) = 0$. Therefore $E^2_{0,2} = H_0((A_*, \partial)) = 0$.

- $E^2_{1,1} = H_1H_1(\mathcal{C}_\bullet(G))$. The chain complex is now $A_n = H_1(\mathcal{C}_n(G))$. The differential $\partial\colon A_1 \to A_0$ is zero, since d_0 and d_1 are constant maps at simplicial level 1. We claim that $\partial\colon A_2 \to A_1$ is zero. To see this, we analyze the effect in H_1 of the respective face maps $d_i\colon G \times G \to G$ for the nerve of G at simplicial level 2. For $i = 0, 2$ the induced homomorphisms are just the projections on the first and second coordinate respectively. For $i = 1$, d_1 is the product in G, so $(d_1)_*$ is the addition map in $H_1(G)$. Therefore the alternating sum of $(d_i)_*$'s is zero. Now then, the differential ∂ factors through

$$H_1(\mathcal{C}_2(G)) \to H_1(G \times G) \xrightarrow{0} H_1(G)$$

and now we see that ∂ must be the zero homomorphism. Therefore $E^2_{1,1} = H_1(G)$.

- The term $E^2_{2,0} = H_2H_0(\mathcal{C}_\bullet(G))$ by definition.

To finish our proof we need to show that $E^2_{p,q} = E^\infty_{p,q}$ when $p + q = 2$ and that there are no extension problems. The differential $d_2\colon E^2_{2,0} \to E^2_{0,1} = 0$ must be zero and thus $E^2_{2,0}$ survives in the E^∞-page. To see that $E^2_{1,1} = E^\infty_{1,1}$, we will show that the differential $d_2\colon E^2_{3,0} \to E^2_{1,1}$ is zero. Consider the simplicial map arising by inclusions $\iota_\bullet\colon \mathcal{C}_\bullet(G) \to NG$. Let $\mathbb{E}^r_{p,q}$ denote the spectral sequence associated to the nerve NG. Then $\iota_\bullet$ induces a map of spectral sequences $\iota_*\colon E^r_{p,q} \to \mathbb{E}^r_{p,q}$. Notice that $\iota_1\colon \mathcal{C}_1(G) = G \to G$ is the identity map which itself induces the identity isomorphism $E^2_{1,1} = H_1(H_1(\mathcal{C}_\bullet(G))) \to \mathbb{E}^2_{1,1} = H_1(H_1(NG))$. Since ι_* commutes with d_2 it is enough to see that $d_2\colon \mathbb{E}^2_{3,0} \to \mathbb{E}^2_{1,1}$ is zero. Since $\mathbb{E}^2_{3,0} = H_3(H_0(NG))$ and G is connected it follows that $(H_0(NG), \partial)$ is an exact chain complex. Thus $\mathbb{E}^2_{3,0} = 0$ and d_2 is zero.

Now we verify that there are no extension problems. We have that $H_2(BG) \cong \mathbb{E}^2_{1,1}$ (clearly $H_0(H_2(NG)) = 0$ and $H_2(H_0(NG)) = 0$ by similar arguments as before). Then $\iota_\bullet$ induces a morphism of extensions

$$\begin{tikzcd}
0 \arrow[r] & E^2_{1,1} \arrow[r] \arrow[d, equal] & H_2(B_{\mathrm{com}}G) \arrow[r] \arrow[d, "\iota_*"] & E^2_{2,0} \arrow[r] & 0 \\
& \mathbb{E}^2_{1,1} \arrow[r, "\cong"] & H_2(BG), & &
\end{tikzcd}$$

and by commutativity of the diagram, $E^2_{1,1} \to H_2(B_{\mathrm{com}}G)$ splits. $\square$

Lemma 4.3. *$H_2(B_{\mathrm{com}}SO(3); \mathbb{Z}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$.*

Proof. The group $H_1(SO(3); \mathbb{Z}) = \mathbb{Z}/2$ so that by Lemma 4.2 we only need to show that the group $H_2H_0(\mathcal{C}_\bullet(SO(3))) = \mathbb{Z}/2$. This case is more interesting, since

TABLE 1. ∂ restricted to $H_0(S^3/Q_8)$ with zero image

	(c_1,c_2,I)	(c_1,c_2,c_1)	(I,c_2,c_3)	(c_1,I,c_3)	(c_1,c_2,c_3)
$(d_0)_*$	$(1,0)$	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$
$(d_1)_*$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$
$(d_2)_*$	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(1,0)$
$(d_3)_*$	$(0,1)$	$(0,1)$	$(1,0)$	$(1,0)$	$(0,1)$

$\mathcal{C}_n(SO(3))$ is not path connected for $n \geq 2$ (see for example [20, Theorem 2.4]). We follow the notation of Lemma 4.2. We are interested in the sequence

$$A_3 = H_0(\mathcal{C}_3(SO(3))) \xrightarrow{\partial_2} A_2 = H_0(\mathcal{C}_2(SO(3))) \xrightarrow{\partial_1} A_1 = H_0(SO(3)).$$

By [20, Theorem 2.4], $\mathcal{C}_2(SO(3))$ has 2 connected components and $\mathcal{C}_3(SO(3))$ has 8. Let $\mathrm{Hom}(\mathbb{Z}^n, SO(3))_{\mathbb{1}}$ denote the connected component of $\mathcal{C}_n(SO(3))$ that contains the trivial representation $\mathbb{1}\colon \mathbb{Z}^n \to SO(3)$ (which is represented by the n-tuple $(I,...,I)$). The connected components of $\mathcal{C}_n(SO(3))$ are either $\mathrm{Hom}(\mathbb{Z}^n, SO(3))_{\mathbb{1}}$ or homeomorphic to S^3/Q_8. The face maps restrict to the components containing $\mathbb{1}$, that is, $d_i\colon \mathrm{Hom}(\mathbb{Z}^n, SO(3))_{\mathbb{1}} \to \mathrm{Hom}(\mathbb{Z}^{n-1}, SO(3))_{\mathbb{1}}$. Since ∂_1 restricted to the homology of each component is the alternating sum of three identity homomorphisms, the differential ∂_1 sends both generators $(1,0)$ and $(0,1)$ in

$$A_2 = H_0(\mathrm{Hom}(\mathbb{Z}^2, SO(3))_{\mathbb{1}}) \oplus H_0(S^3/Q_8) \cong \mathbb{Z} \oplus \mathbb{Z}$$

to the generator in $A_1 = \mathbb{Z}$. Thus $\ker \partial_1 = \langle(-1,1)\rangle$.

Now then, the differential ∂_2 restricted to the summand $H_0(\mathrm{Hom}(\mathbb{Z}^3, SO(3))_{\mathbb{1}})$ of A_3 is the alternating sum of 4 identity homomorphisms which is then the zero homomorphism. Now we analyze the values of the face maps on the the connected components that do not contain $\mathbb{1}$. Each such component has the form $\{g(x_1,x_2,x_3)g^{-1} : g \in SO(3)\}$ for some 3–tuple $(x_1,x_2,x_3) \in SO(3)^3$ with $\langle x_1,x_2,x_3\rangle \cong D_4 = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ (this follows from the arguments in [6, Section 3]). To write canonical representatives of these components, choose elements distinct, non-identity elements $c_1,c_2,c_3 \in SO(3)$ such that $\langle c_1,c_2,c_3\rangle \cong D_4$ (so $c_3 = c_1c_2$). The action of the quotient group $N_{SO(3)}(D_4)/D_4 \cong \Sigma_3$ by conjugation on D_4 is by permuting the elements c_j. The 7 remaining components in $\mathcal{C}_3(SO(3))$ (all homeomorphic to S^3/Q_8) are then represented by the 3-tuples

$$(c_1,c_2,I),(c_1,c_2,c_2),(I,c_2,c_3),(c_2,c_2,c_3),(c_1,I,c_3),(c_1,c_2,c_1),(c_1,c_2,c_3)$$

To compute the values of $(d_i)_*$ on these components, we only need to know the connected component of $\mathcal{C}_2(SO(3))$ where the image of d_i lands. For our purposes let us write the obvious identities $c_1c_2 = c_3$, $c_1c_3 = c_2$ and $c_2c_3 = c_1$ in D_4. Using that the pairs of the form (I,c_j), (c_j,I) and (c_j,c_j) are in $\mathrm{Hom}(\mathbb{Z}^2, SO(3))_{\mathbb{1}}$ for $1 \leq j \leq 3$, it is easy to check that the alternating sum of $(d_i)_*$, with $0 \leq i \leq 3$ at each of the 5 components $H_0(S^3/Q_8)$ represented by $(c_1,c_2,I),(I,c_2,c_3),(c_1,I,c_3),(c_1,c_2,c_1)$ and (c_1,c_2,c_3) is zero. For the ones represented by (c_1,c_2,c_2) and (c_2,c_2,c_3), the image of the alternating sum of $(d_i)_*$ is the ideal generated by $(-2,2)$. We include both computations in Tables 1 and 2, where it can be seen that the alternating sum of $(d_i)_*$ is as claimed. We conclude that $E^2_{2,0} = \langle(-1,1)\rangle/\langle(-2,2)\rangle = \mathbb{Z}/2$. □

Proposition 4.4.

TABLE 2. ∂ restricted to $H_0(S^3/Q_8)$ with non-zero image

	(c_1, c_2, c_2)	(c_2, c_2, c_3)
$(d_0)_*$	$(1,0)$	$(0,1)$
$(d_1)_*$	$(0,1)$	$(1,0)$
$(d_2)_*$	$(1,0)$	$(0,1)$
$(d_3)_*$	$(0,1)$	$(1,0)$

(1) *$\pi_2(B_{\text{com}}SO(3)) = \pi_2(B_{\text{com}}O(3)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$;*
(2) *For $n \geq 2$, the inclusions $B_{\text{com}}O(n) \to B_{\text{com}}O(n+1)$ are 2-connected and for $n \geq 3$ they induce isomorphisms in π_2, and*
(3) *For any $n \geq 3$, the inclusions $B_{\text{com}}SO(n) \to B_{\text{com}}SO(n+1)$ induce isomorphisms in π_2.*

Proof. (1) Since $SO(3)$ is path-connected, $B_{\text{com}}SO(3)$ is simply connected. By the Hurewicz Isomorphism Theorem and Lemma 4.3, $\pi_2(B_{\text{com}}SO(3)) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. Recall that $O(3) \cong \{\pm I\} \times SO(3)$, so that $B_{\text{com}}O(3) \cong B\{\pm I\} \times B_{\text{com}}SO(3)$. Since $B\{\pm I\}$ is an Eilenberg-MacLane space of type $K(\mathbb{Z}/2, 1)$, we see that

$$\pi_2(B_{\text{com}}O(3)) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2. \tag{6}$$

(2)-(3) We claim that for any $n \geq 3$ the inclusions $B_{\text{com}}O(n) \to B_{\text{com}}O(n+1)$ are 2-connected (note, in particular, that this implies surjectivity at the level of π_2). We have already mentioned that $C_\bullet(G)$ is a proper simplicial space for any Lie group G, and by [11, Appendix A], the *fat* geometric realization $\|C_\bullet(G)\|$ is equivalent to $B_{\text{com}}G$. Then by [7, Lemma 2.4] to prove our claim for the orthogonal groups, we only need to show that $C_k(O(n)) \to C_k(O(n+1))$ is $2-k$ connected for $k = 0, 1$ and 2. The case $k = 0$ is trivial since $C_0(G) = \{pt\}$. For $k = 1$, it is well known that the inclusions $O(n) \to O(n+1)$ induce isomorphisms in fundamental groups and bijections at the level of π_0. By [10, Theorem 1.1] we have a bijection $\pi_0(C_2(O(n))) \to \pi_0(C_2(O(n+1)))$ whenever $n \geq 2^2 - 1 = 3$. Therefore $\pi_2(B_{\text{com}}O(n)) \to \pi_2(B_{\text{com}}O(n+1))$ is surjective.

Similarly, for every $n \geq 3$, using the well known behavior of the inclusions $SO(n) \to SO(n+1)$ in π_1 and [10, Corollary 1.5] we see that

$$B_{\text{com}}SO(n) \longrightarrow B_{\text{com}}SO(n+1)$$

is also 2-connected. Therefore $\pi_2(B_{\text{com}}SO(n)) \to \pi_2(B_{\text{com}}SO(n+1))$ is surjective as well.

To prove part (2), first notice that part (1) and Proposition 4.1 for $n = 3$ imply that $\pi_2(B_{\text{com}}O(2)) \to \pi_2(B_{\text{com}}O(3))$ is surjective. The isomorphism (6) and surjectivity at π_2 of the inclusions $B_{\text{com}}O(n) \to B_{\text{com}}O(n+1)$ imply that the groups $\pi_2(B_{\text{com}}O(n))$ are all quotients of $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. By Proposition 4.1, for any $n \geq 3$ we see two distinct non-trivial group elements in $\pi_2(B_{\text{com}}O(n))$ (e.g. $[j_n \circ f_1]$ and $[j_n \circ (j \circ g_1)]$ as in the notation of Theorem 3.9 and Proposition 4.1). Thus the only possible quotient is $\mathbb{Z}/2 \oplus \mathbb{Z}/2$, and then the isomorphisms in part (2) follow.

Naturality of the isomorphisms $\{\pm I\} \times SO(2n+1) \cong O(2n+1)$ with respect to the aforementioned inclusions and the induced isomorphisms at the level of π_2 also give the claimed isomorphisms in part (3). $\square$

Remark 4.5. The inclusions in Proposition 4.4 are actually 3-connected whenever $n \geq 7$. This follows from earlier work done with spaces of commuting elements in Lie groups. We give a sketch proof of this claim by showing that $C_k(O(n)) \to C_k(O(n+1))$ is $3-k$-connected for $k = 0, 1, 2, 3$ and $n \geq 7$. Again, $k = 0$ follows trivially and for $k = 1$ we use that $\pi_2(G) = 0$, and thus the inclusions are 2-connected. For $k = 2$ it would remain to show that $\mathrm{Hom}(\mathbb{Z}^2, O(n))_{\mathbb{1}} \to \mathrm{Hom}(\mathbb{Z}^2, O(n+1))_{\mathbb{1}}$ is surjective in π_1, but this follows from [8, Theorem 1.1], which states that $\pi_1(C_k(G), \mathbb{1}) \cong \pi_1(G)^k$ for compact Lie groups and the map that witnesses this isomorphism is natural with respect to G (see [8, p. 41]). For $k = 3$, the statement follows form [10, Theorem 1.1 and Corollary 1.5] where $n \geq 2^3 - 1 = 7$. Similarly for the special orthogonal groups.

Corollary 4.6. *For every $n \geq 2$, the inclusions $B_{\mathrm{com}}O(n) \to B_{\mathrm{com}}O(n+1)$ induce isomorphisms $H^2(B_{\mathrm{com}}O(n+1); \mathbb{F}_2) \xrightarrow{\cong} H^2(B_{\mathrm{com}}O(n); \mathbb{F}_2)$.*

Proof. Part (2) of Proposition 4.4 and the Serre exact sequence (McCleary [14, Example 5.D]) of the homotopy fibration $F \to B_{\mathrm{com}}O(n) \to B_{\mathrm{com}}O(n+1)$ imply that

$$H^2(B_{\mathrm{com}}O(n+1); \mathbb{F}_2) \to H^2(B_{\mathrm{com}}O(n); \mathbb{F}_2)$$

is injective. It remains to show that all these groups have the same rank. Since $B_{\mathrm{com}}SO(n)$ is simply connected, to compute its $\mathbb{F}_2$-cohomology for $n \geq 3$ we can use parts (1) and (3) of Proposition 4.4, and together with the Künneth formula we see that

$$\begin{aligned} H^2(B_{\mathrm{com}}O(2n+1); \mathbb{F}_2) &\cong H^2(B_{\mathrm{com}}SO(2n+1); \mathbb{F}_2) \oplus H^2(\mathbb{RP}^\infty; \mathbb{F}_2) \\ &\cong (\mathbb{F}_2 \oplus \mathbb{F}_2) \oplus \mathbb{F}_2. \end{aligned}$$

Hence for $n \geq 3$ and odd, the groups $H^2(B_{\mathrm{com}}O(n); \mathbb{F}_2)$ are rank 3. Injectivity for all $n \geq 3$ implies now that $H^2(B_{\mathrm{com}}O(n+1); \mathbb{F}_2) \to H^2(B_{\mathrm{com}}O(n); \mathbb{F}_2)$ are isomorphisms. By [5, Theorem 5.1], $H^2(B_{\mathrm{com}}O(2); \mathbb{F}_2)$ has rank 3 and thus $H^2(B_{\mathrm{com}}O(3); \mathbb{F}_2) \to H^2(B_{\mathrm{com}}O(2); \mathbb{F}_2)$ is an isomorphism as well. □

Corollary 4.6 allows us to define a new characteristic class a_2 for commutative bundles. We will use this class in several subsequent arguments.

Definition 4.7. Let $\bar{r} \in H^2(B_{\mathrm{com}}O(2); \mathbb{F}_2)$ denote the reduction mod 2 of $r \in H^2(B_{\mathrm{com}}O(2); \mathbb{Z})$ (as in (5)). By Corollary 4.6, there exists a unique class $a_2 \in H^2(B_{\mathrm{com}}O; \mathbb{F}_2)$ satisfying $i^*(a_2) = \bar{r}$, where $i\colon B_{\mathrm{com}}O(2) \hookrightarrow B_{\mathrm{com}}O$ is the inclusion. (The restriction of a_2 to a class on $B_{\mathrm{com}}O(n)$ will also be denoted by a_2.)

4.3. **Additive structure of $\widetilde{KO}_{\mathrm{com}}(\Sigma)$.** Gritschacher, using very different methods from those in the present paper, established an isomorphism

$$\widetilde{KO}_{\mathrm{com}}(S^2) \cong \mathbb{Z}/2 \oplus KO(S^2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

Using Proposition 4.4, we now give a new proof of this isomorphism, and identify generators for this group arising from our transitionally commutative $O(2)$–bundles.

Let $E_0^{f_1}$ denote the trivial real vector bundle of rank 2 over S^2 with transitionally commutative structure f_1. Let E_1 be the oriented vector bundle classified by the generator $1 \in \pi_2(BO(2)) \cong \mathbb{Z}$, and let E_1^{st} denote E_1 with its algebraic TC structure $S^2 \xrightarrow{g_1} BSO(2) \xrightarrow{j} B_{\mathrm{com}}O(2)$.

Proposition 4.8. *The commutative K–theory classes associated to $E_0^{f_1}$ and E_1^{st} generate the group*

$$\widetilde{KO}_{\mathrm{com}}(S^2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

Proof. By Proposition 4.4, $\pi_2(B_{\mathrm{com}}O) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. Since $B_{\mathrm{com}}O$ is a (connected) H–space under block sum of commuting matrices, we have an isomorphism

$$\widetilde{KO}_{\mathrm{com}}(S^2) = [S^2, B_{\mathrm{com}}O] \cong \pi_2(B_{\mathrm{com}}O).$$

Next, note that any two distinct non-zero elements of $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ generate this group, so it suffices to show that the classes associated to $E_0^{f_1}$ and E_1^{st} are non-zero and distinct. The class associated to $E_0^{f_1}$ is non-zero by Proposition 4.1. The class associated to E_1^{st} is non-zero since

$$S^2 \xrightarrow{g_1} BSO(2) \hookrightarrow BO(2)$$

is a generator of $\pi_2(BO(2))$, and the map $\pi_2(BO(2)) \to \pi_2(BO)$ is surjective. Since the composition $S^2 \xrightarrow{f_1} B_{\mathrm{com}}O(2) \hookrightarrow BO(2)$ is null-homotopic, these classes are distinct. □

Let $i\colon B_{\mathrm{com}}O(2) \to B_{\mathrm{com}}O$ be the inclusion, and for a closed connected surface Σ, let $c\colon \Sigma \to S^2$ denote the map that collapses the 1-skeleton of the single 2-cell decomposition of Σ.

Theorem 4.9. *Let Σ be a closed connected surface. Then*

$$\widetilde{KO}_{\mathrm{com}}(\Sigma) \cong \widetilde{KO}(\Sigma) \oplus \mathbb{Z}/2,$$

and the kernel of $\iota_*\colon \widetilde{KO}_{\mathrm{com}}(\Sigma) \to \widetilde{KO}(\Sigma)$ is generated by $i \circ f_1 \circ c\colon \Sigma \to B_{\mathrm{com}}O$, and $(i \circ f_1 \circ c)^*(a_2)$ is the generator in $H^2(\Sigma; \mathbb{F}_2)$.

Proof. As shown in the Appendix, we have a natural isomorphism

$$\ker(\iota_*) \cong [\Sigma, E_{\mathrm{com}}O].$$

Consider the cofibration sequence $\bigvee_m S^1 \to \Sigma \xrightarrow{c} S^2$, which induces the sequence

$$[S^2, E_{\mathrm{com}}O] \xrightarrow{-\circ c} [\Sigma, E_{\mathrm{com}}O] \to \left[\bigvee_m S^1, E_{\mathrm{com}}O\right].$$

Since the map $\pi_2(B_{\mathrm{com}}O) \to \pi_2(BO)$ induced by ι is surjective and $\pi_1(B_{\mathrm{com}}O) \cong \pi_1(BO)$ (see [4, Lemma 4.3]), $E_{\mathrm{com}}O$ is simply connected, and then the last term in the above sequence is trivial. Therefore $[S^2, E_{\mathrm{com}}O] \xrightarrow{-\circ c} [\Sigma, E_{\mathrm{com}}O]$ is surjective.

Taking $\Sigma = S^2$, the proof of Proposition 4.8 shows that

$$\iota_*\colon \widetilde{KO}_{\mathrm{com}}(S^2) \to \widetilde{KO}(S^2)$$

is a surjection $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \mathbb{Z}/2$, so $\ker(\iota_*) \cong \mathbb{Z}/2$. Moreover, as above we have $\ker(\iota_*) \cong [S^2, E_{\mathrm{com}}O]$. It follows that $[\Sigma, E_{\mathrm{com}}O]$ has at most two elements, and to complete the proof, it suffices to show that $[i \circ f_1 \circ c]$ is a non-trivial element of $\ker(\iota_*) \cong [\Sigma, E_{\mathrm{com}}O]$.

By Lemma 3.2, we have $[i \circ f_1 \circ c] \in \ker(\iota_*)$. To show that $[i \circ f_1 \circ c]$ is non-trivial, we study the values of the composition

$$\Sigma \xrightarrow{c} S^2 \xrightarrow{f_1} B_{\mathrm{com}}O(2) \xrightarrow{i} B_{\mathrm{com}}O$$

in $\mathbb{F}_2$-cohomology. Recall from Definition 4.7 that we have a class

$$a_2 \in H^2(B_{\text{com}}O;\mathbb{F}_2)$$

that pulls back to $\bar{r}$ (the reduction mod 2 of $r \in H^2(B_{\text{com}}O(2);\mathbb{Z})$). By Lemma 3.7, $f_1^*(\bar{r}) \in H^2(S^2;\mathbb{F}_2)$ is the generator. Finally. since $c^*\colon H^2(S^2;\mathbb{F}_2) \to H^*(\Sigma;\mathbb{F}_2)$ is an isomorphism, $(i \circ f_1 \circ c)^*(a_2)$ is the generator in $H^2(\Sigma;\mathbb{F}_2)$. □

Remark 4.10. We note that Theorem 4.9 can be proven without referring to the Appendix. The above argument shows that $\ker(\iota_*)$ contains at least two elements, and that $[\Sigma, E_{\text{com}}O]$ has at most two elements. But the homotopy lifting property for the homotopy fibration $E_{\text{com}}O \to B_{\text{com}}O \xrightarrow{\iota} BO$ yields a surjection $[\Sigma, E_{\text{com}}O] \twoheadrightarrow \ker(\iota_*)$.

5. Real commutative K-theory of surfaces: Multiplicative structure

In this section we compute the ring structure of $\widetilde{KO}_{\text{com}}(\Sigma)$ for closed, connected surfaces Σ. This will be achieved using the characteristic class a_2 introduced in Definition 4.7. In order to work with this class, we will need some results regarding the effect of

$$\phi^{-1}\colon B_{\text{com}}O(2) \to B_{\text{com}}O(2)$$

in cohomology with $\mathbb{F}_2$–coefficients.

5.1. **The inverse map in $\mathbb{F}_2$ cohomology.**

In [5, Theorem 5.1], a presentation

$$H^*(B_{\text{com}}O(2);\mathbb{F}_2) \cong \mathbb{F}_2[w_1, w_2, \bar{r}, s]/(w_1\bar{r}, w_1 s, \bar{r}^2, s^2)$$

is given, where w_1 and w_2 are the pullbacks of the first and second Stiefel–Whitney classes along the inclusion $\iota\colon B_{\text{com}}O(2) \to BO(2)$; the class $\bar{r}$ is the reduction of the class $r \in H^2(B_{\text{com}}O(2);\mathbb{Z})$ described in (5); and $s \in H^3(B_{\text{com}}O(2);\mathbb{F}_2)$. It is also shown there that the action of the Steenrod algebra is determined by its action on $H^*(BO(2);\mathbb{F}_2)$ and the total Steenrod squares $\text{Sq}(\bar{r}) = \bar{r}$ and

$$\text{Sq}(s) = s + w_2\bar{r} + w_1^2 s \tag{7}$$

Proposition 5.1. *$(\phi^{-1})^*\colon H^*(B_{\text{com}}O(2);\mathbb{F}_2) \to H^*(B_{\text{com}}O(2);\mathbb{F}_2)$ is given on generators by $w_1 \mapsto w_1$, $w_2 \mapsto w_2 + \bar{r}$, $\bar{r} \mapsto \bar{r}$ and $s \mapsto s$.*

Proof. We study the case of each class one at a time.

- $w_1 \mapsto w_1$. Since $(\phi^{-1})^*$ is an involution, it follows that $(\phi^{-1})^*(w_1)$ is not zero and hence $(\phi^{-1})^*(w_1) = w_1$.
- $\bar{r} \mapsto \bar{r}$. To compute the image of $\bar{r}$, first notice that $\phi^{-1} \circ j = j \circ \phi^{-1}$. By Lemma 3.5 the restriction of $(\phi^{-1})^*$ to $BSO(2)$ is multiplication by -1, and by relation 5, we see that integrally $\phi^{-1}(r) = -r$. Therefore $(\phi^{-1})^*(\bar{r}) = \bar{r}$.
- $w_2 \mapsto w_2 + \bar{r}$. We write $(\phi^{-1})^*(w_2)$ as a generic element $(\phi^{-1})^*(w_2) = \epsilon_1 w_1^2 + \epsilon_2 w_2 + \epsilon_3 \bar{r}$ with $\epsilon_i \in \{0, 1\}$. To show that ϵ_3 is non-zero, recall that Proposition 3.4 for $k = -1$ and $n = -1$ says that $\iota \circ \phi^{-1} \circ f_{-1}$ has degree 1, and in particular, $f_{-1}^*(\phi^{-1})^*\iota^*(w_2)$ is the generator $\tau \in H^2(S^2;\mathbb{F}_2)$. Therefore

$$\tau = f_{-1}^*(\phi^{-1})^*\iota^*(w_2) = f_{-1}^*(\phi^{-1})^*(w_2) = f_{-1}^*(\epsilon_1 w_1^2 + \epsilon_2 w_2 + \epsilon_3\bar{r})$$

Now then, 3.4 for $n = 1$ implies that $\iota \circ f_k$ is homotopy trivial, and for $i = 1, 2$, we see that $f_{-1}^*(w_i) = 0$. Therefore $(\phi^{-1})^*(w_2)$ must have the class $\bar{r}$ and thus $\epsilon_3 = 1$.

For the value of ϵ_2, once more, using that ϕ^{-1} is an involution we see the equation

$$\begin{aligned} w_2 &= (\phi^{-1})^*((\phi^{-1})^*(w_2)) \\ &= (\phi^{-1})^*(\epsilon_1 w_1^2 + \epsilon_2 w_2 + \bar{r}) \\ &= \epsilon_1 w_1^2 + (\epsilon_2\epsilon_1 w_1^2 + \epsilon_2\epsilon_2 w_2 + \epsilon_2 \bar{r}) + \bar{r} \end{aligned}$$

which implies $\epsilon_2 = 1$. Lastly, to show that $\epsilon_1 = 0$, we use the inversion map restricted to the abelian subgroup $O(1)^2 \subset O(2)$. Any element in $O(1) \times O(1)$ has order 2, so that $(-)^{-1}\colon O(1)^2 \to O(1)^2$ is the identity and hence $\phi^{-1}\colon BO(1)^2 \to BO(1)^2$ is the identity as well. Since $O(1)^2$ is abelian, the inclusion $i\colon BO(1)^2 \hookrightarrow BO(2)$ factors through $B_{\mathrm{com}}O(2)$, and we see the commutative diagram

$$\begin{tikzcd} BO(1)^2 \arrow[r, equal, "\phi^{-1}"] \arrow[d, "k"] & BO(1)^2 \arrow[d, "k"] \arrow[dr, bend left, "i"] & \\ B_{\mathrm{com}}O(2) \arrow[r, "\phi^{-1}"] & B_{\mathrm{com}}O(2) \arrow[r, "\iota"] & BO(2). \end{tikzcd}$$

Fix a presentation $H^*(BO(1)^2;\mathbb{F}_2) = \mathbb{F}_2[u,v]$ such that i^* gives the isomorphism with the invariants subring $\mathbb{F}_2[u,v]^{\mathbb{Z}/2} \cong H^*(BO(2);\mathbb{F}_2)$, where the invariants are taken under the action of permuting u and v, that is, $i^*(w_1) = u+v$ and $i^*(w_2) = uv$. Applying $H^*(-;\mathbb{F}_2)$ to the diagram, we see that $(\phi^{-1})^* i^*(w_2) = uv$ must equal $k^*(\phi^{-1})^*\iota^*(w_2) = k^*(\epsilon_1 w_1^2 + w_2 + \bar{r}) = \epsilon_1(u^2+v^2) + uv + k^*(\bar{r})$, where the last equality holds by commutativity of the triangle in the diagram. It is shown in [5, p 23] that $k^*(r) = 0$, so we can conclude $\epsilon_1 = 0$.

- $s \mapsto s$. Following the same procedure as above, let us write $(\phi^{-1})^*(s) = \epsilon_1 w_1^3 + \epsilon_2 w_1 w_2 + \epsilon_3 s$ with $\epsilon_i \in \{0,1\}$. Similarly, ϕ^{-1} being an involution gives $\epsilon_3 = 1$. To analyze ϵ_1, we use Sq^1 (recall that on $H^*(BO(2);\mathbb{F}_2)$, the total Steenrod squares are $\mathrm{Sq}(w_1) = w_1 + w_1^2$ and $\mathrm{Sq}(w_2) = w_2 + w_1w_2 + w_2^2$). Using equation (7) we see that $\mathrm{Sq}^1(s) = \bar{r}w_2$ and thus

$$(\phi^{-1})^*(\mathrm{Sq}^1(s)) = (\phi^{-1})^*(\bar{r}w_2) = \bar{r}w_2$$

and by naturality of Sq^1 this must be equal to

$$\begin{aligned} \mathrm{Sq}^1((\phi^{-1})^*(s)) &= \mathrm{Sq}^1(\epsilon_1 w_1^3 + \epsilon_2 w_1 w_2 + s) \\ &= \epsilon_1 w_1^4 + \epsilon_2(w_1(w_1w_2) + w_1^2 w_2) + \bar{r}w_2 \\ &= \epsilon_1 w_1^4 + \bar{r}w_2 \end{aligned}$$

giving $\epsilon_1 = 0$. For the value of ϵ_2 we use Sq^2. From equation (7) we get $\mathrm{Sq}^2(s) = w_1^2 s$. On one side we have

$$(\phi^{-1})^*(\mathrm{Sq}^2(s)) = (\phi^{-1})^*(w_1^2 s) = w_1^2(\epsilon_2 w_1 w_2 + s) = \epsilon_2 w_1^3 w_2 + w_1^2 s$$

which must equal $\mathrm{Sq}^2(\phi^{-1})^*(s) = \mathrm{Sq}^2(\epsilon_2 w_1 w_2 + s) = \epsilon_2(w_1^3 w_2 + w_1 w_2^2) + w_1^2 s$. Then the only possible choice is $\epsilon_2 = 0$. □

Remark 5.2. Recall that the inclusion $BO(1)^n \to BO(n)$ induces an injective map in cohomology with $\mathbb{F}_2$-coefficients, and since $O(1)^n$ is abelian, the inclusion factors through $\iota\colon B_{\mathrm{com}}O(n) \to BO(n)$. This yields a commutative diagram

$$\begin{tikzcd} H^*(BO(n);\mathbb{F}_2) \arrow[r, "\iota^*"] \arrow[dr] & H^*(B_{\mathrm{com}}O(n);\mathbb{F}_2) \arrow[d] \\ & H^*(BO(1)^n;\mathbb{F}_2) \end{tikzcd},$$

and thus ι^* is injective (by the Splitting Principle). Then for any $1 \le i \le n$ we can define Stiefel–Whitney classes in $H^*(B_{\mathrm{com}}O(n);\mathbb{F}_2)$ as the pullback under ι of the ordinary Stiefel–Whitney classes $w_i \in H^i(BO(n);\mathbb{F}_2)$; that is

$$w_i := \iota^*(w_i) \in H^i(B_{\mathrm{com}}O(n);\mathbb{F}_2).$$

Definition 4.7 provides a class $a_2 \in H^2(B_{\mathrm{com}}O(n);\mathbb{F}_2)$ such that $j_n^*(a_2) = \bar{r} \in H^2(B_{\mathrm{com}}O(2);\mathbb{F}_2)$. Consider $w_2 \in H^2(B_{\mathrm{com}}O(n);\mathbb{F}_2)$. Then 5.1 implies that

$$(\phi^{-1})^*(w_2) = w_2 + a_2,$$

and the analogous formula holds in $H^2(B_{\mathrm{com}}O;\mathbb{F}_2)$ as well.

Remark 5.3. (*Obstruction to algebraic TC structures*) Given a TC structure $f\colon X \to B_{\mathrm{com}}O(n)$, it is interesting to ask whether f is algebraic. Recall that this amounts to asking whether f factors (up to homotopy) through the inclusion $i_A\colon BA \hookrightarrow B_{\mathrm{com}}O(n)$ associated to an abelian subgroup $A \le O(n)$.

We claim that the class a_2 is an obstruction to algebraicity. It can be shown that every abelian subgroup $A \subset O(n)$ is conjugate to a subgroup of the product $O(1)^{n-2k} \times SO(2)^k$ (thought of as subgroup of block–diagonal matrices in $O(n)$). Hence we may assume, without loss of generality, that $A = O(1)^{n-2k} \times SO(2)^k$. Consider a map $g\colon X \to BA$. Then

$$(i_A \circ g)^*(a_2) = (i_A \circ g)^*(w_2) + (\phi^{-1} \circ i_A \circ g)^*(w_2).$$

Now $\phi^{-1} \circ i_A = i_A \circ \phi^{-1}$, and

$$(\phi^{-1})^*\colon H^2(BA;\mathbb{F}_2) \to H^2(BA;\mathbb{F}_2)$$

is the identity, since $\phi^{-1}\colon O(1) \to O(1)$ is the identity and Lemma 3.5 implies that $(\phi^{-1})^*\colon H^2(BSO(2);\mathbb{F}_2) \to H^2(BSO(2);\mathbb{F}_2)$ is the identity. It now follows that $(\phi^{-1} \circ i_A)^* = i_A^*$ and then $(i_A \circ g)^*(a_2) = 0$. In other words, if E is a bundle with an algebraic TC structure f, then $a_2(E^f) = 0$.

5.2. **Ring structure of $\widetilde{KO}_{\mathrm{com}}(\Sigma)$.** Block sum of matrices and Kronecker product define homomorphisms $\oplus\colon O(n) \times O(m) \to O(n+m)$ and $\otimes\colon O(n) \times O(m) \to O(mn)$, which then induce maps $B_{\mathrm{com}}(\oplus)$ and $B_{\mathrm{com}}(\otimes)$. These two maps endow the set of homotopy classes of maps $[X, \coprod_{n=0}^{\infty} B_{\mathrm{com}}O(n)]$ with a *semi-ring* structure. Topological commutative real K-theory can then be defined as the Grothendieck ring

$$KO_{\mathrm{com}}(X) := \mathrm{Gr}\left[X, \coprod_{n=0}^{\infty} B_{\mathrm{com}}O(n)\right].$$

For $n \ge 0$ the inclusions $B_{\mathrm{com}}O(n) \to B_{\mathrm{com}}O$ induce an isomorphism of rings $KO_{\mathrm{com}}(X) \xrightarrow{\cong} [X, \mathbb{Z} \times B_{\mathrm{com}}O]$ (see [4, Theorem 5.5]). Recall that our definition of reduced commutative K-theory is $\widetilde{KO}_{\mathrm{com}}(X) = [X, B_{\mathrm{com}}O]$, which now under this isomorphism, can be identified with the non-unital ring generated by the formal differences $[f] - [*_n]$, where $*_n, f\colon X \to B_{\mathrm{com}}O(n)$ and $*_n$ is the null-map. For our convenience we will represent these classes as formal differences of vector bundles with a fixed TC structure. Let $(\varepsilon^n)^{\mathrm{st}}$ denote the trivial bundle ε^n with TC structure given by $*_n$, and let E^f denote a rank n vector bundle E with TC structure f (the underlying bundle E is classified by the map $\iota \circ f\colon X \to BO(n)$). Then the stable class E^f (or $[f]$) in $\widetilde{KO}_{\mathrm{com}}(X)$ is represented by $E^f - (\varepsilon^n)^{\mathrm{st}}$.

Recall that by Theorem 4.9, there is a unique non-zero element in the kernel of the natural map $\widetilde{KO}_{\mathrm{com}}(\Sigma) \to \widetilde{KO}(\Sigma)$. We refer to this element as *the non-standard stable class*. As an application of Proposition 5.1, we show that for a closed connected surface Σ all products in $\widetilde{KO}_{\mathrm{com}}(\Sigma)$ with the non-standard stable class are trivial. Our calculations will use the next two formulas, which may be verified using the splitting principle. Let E and F be rank 2 real vector bundles, and let L be a real line bundle, all over the same base space. Then

$$w_2(E \otimes F) = w_1(E)^2 + w_1(E)w_1(F) + w_1(F)^2 \tag{8}$$

$$w_2(E \otimes L) = w_1(E)^2 + w_1(L)^2 + w_2(E). \tag{9}$$

Lemma 5.4. *For any $x \in \widetilde{KO}_{\mathrm{com}}(S^2)$, we have $x \cdot (E_0^{f_1} - (\varepsilon^2)^{\mathrm{st}}) = 0$.*

Proof. By Proposition 4.8 we only need to show that the products $(E_0^{f_1} - (\varepsilon^2)^{\mathrm{st}})^2$ and $(E_1^{\mathrm{st}} - (\varepsilon^2)^{\mathrm{st}})(E_0^{f_1} - (\varepsilon^2)^{\mathrm{st}})$ are zero, where the standard (algebraic) TC structure on E_1 is given by $j \circ g_1$. To do this, we classify the elements in $\widetilde{KO}_{\mathrm{com}}(S^2)$ via characteristic classes.

Consider the natural map

$$\pi_2(B_{\mathrm{com}}O) \longrightarrow \mathrm{Hom}(H^2(B_{\mathrm{com}}O;\mathbb{F}_2), H^2(S^2;\mathbb{F}_2)) = \mathrm{Hom}(H^2(B_{\mathrm{com}}O;\mathbb{F}_2), \mathbb{F}_2) \tag{10}$$

sending a class $[f] \in \pi_2(B_{\mathrm{com}}O)$ to the induced map f^* on cohomology (this map is a homomorphism by the Eckmann–Hilton argument). We claim that this map is in fact injective. Note that by Proposition 4.4, we have $\pi_2(B_{\mathrm{com}}O) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, so in particular $\pi_2(B_{\mathrm{com}}O)$ is an $\mathbb{F}_2$–vector space. By Corollary 4.6, the classes w_2 and a_2 form a basis for $H^2(B_{\mathrm{com}}O;\mathbb{F}_2)$, and hence the group $\mathrm{Hom}(H^2(B_{\mathrm{com}}O;\mathbb{F}_2), \mathbb{F}_2)$ is generated by the $\mathbb{F}_2$–duals $w_2^\dagger$ and $a_2^\dagger$. Recall that the representatives of the generators of $\pi_2(B_{\mathrm{com}}O)$ in Proposition 4.8 are $i \circ j \circ g_1$ and $i \circ f_1$ (where $i\colon B_{\mathrm{com}}O(2) \to B_{\mathrm{com}}O$ is the inclusion as before). Then by Lemma 3.7 and equation (5) in $\mathbb{F}_2$-cohomology, we have $(i \circ j \circ g_1)^* = w_2^\dagger$ and $(i \circ f_1)^* = a_2^\dagger$. So the generators of $\pi_2(B_{\mathrm{com}}O)$ have linearly independent images under the map (10), which proves that this map is injective. Note that this implies that the classes E^f in $\widetilde{KO}_{\mathrm{com}}(S^2)$ are completely determined by the values of $w_2(E^f) := w_2(E)$ and $a_2(E^f)$.

We now compute the relevant products in $\widetilde{KO}_{\mathrm{com}}(S^2)$.

- $(E_0^{f_1} - (\varepsilon^2)^{\mathrm{st}})^2 = 0$. This equivalent to $(E_0^{f_1} \otimes E_0^{f_1}) \oplus (\varepsilon^4)^{\mathrm{st}} = 4E_0^{f_1}$. The above characterization says that the equality holds if and only if both sides have the same values on w_2 and a_2. The underlying vector bundle on each side of the equality is a trivial bundle, and thus both classes have vanishing w_2. To compute the values of a_2, recall from Remark 5.2 that

$$a_2(E^f) = w_2(E) + w_2(\phi^{-1}(E^f)),$$

where the TC structure on $\phi^{-1}(E^f)$ is $\phi^{-1} \circ f$ and the underlying bundle is classified by $\iota \circ \phi^{-1} \circ f$. By Proposition 3.4 the map $\iota \circ \phi^{-1} \circ f_1$ is homotopic to the map that classifies a bundle with choice of orientation -1. Hence the underlying bundle of $\phi^{-1}(E_0^{f_1})$ is E_{-1}. Now

$$\begin{aligned} a_2((E_0^{f_1} \otimes E_0^{f_1}) \oplus (\varepsilon^4)^{\mathrm{st}}) &= w_2((\phi^{-1}(E_0^{f_1}) \otimes \phi^{-1}(E_0^{f_1})) \oplus \varepsilon^4) \\ &= w_2(E_{-1} \otimes E_{-1}), \\ &= 0, \end{aligned}$$

where in the last equality we use formula (8) and the fact that $w_1(E_{-1}) = 0$. For the right hand side we see that $a_2(4E_0^{f_1}) = w_2(4\phi^{-1}(E_0^{f_1})) = w_2(4E_{-1}) = 0$.

- $(E_1^{\mathrm{st}} - (\varepsilon^2)^{\mathrm{st}})(E_0^{f_1} - (\varepsilon^2)^{\mathrm{st}}) = 0$. Similarly, this is equivalent to $(E_1^{\mathrm{st}} \otimes E_0^{f_1}) \oplus (\varepsilon^4)^{\mathrm{st}} = 2(E_1^{\mathrm{st}} \oplus E_0^{f_1})$. The underlying vector bundles are $(E_1 \otimes \varepsilon^2) \oplus \varepsilon^4$ and $2(E_1 \oplus \varepsilon^2)$ which are isomorphic (in fact stably trivial) and thus they have the same values on w_2. To compute a_2 we need to determine the underlying bundle of $\phi^{-1}(E_1^{\mathrm{st}}) = \phi^{-1}(E_1^{j\circ g_1})$. This is classified by the degree of the map

$$(\iota \circ \phi^{-1} \circ j \circ g_1)_* = (\iota \circ j \circ \phi^{-1} \circ g_1)_*,$$

which by Lemma 3.5 and the definition of g_1, is -1. Thus the underlying bundle is E_{-1}. Then

$$\begin{aligned} a_2((E_1^{\mathrm{st}} \otimes E_0^{f_1}) \oplus (\varepsilon^4)^{\mathrm{st}}) &= w_2((E_1 \otimes \varepsilon^2) \oplus \varepsilon^4) + w_2((\phi^{-1}(E_1^{\mathrm{st}}) \otimes \phi^{-1}(E_0^{f_1})) \oplus \varepsilon^4) \\ &= w_2(E_{-1} \otimes E_{-1}), \\ &= 0. \end{aligned}$$

Similarly $a_2(2(E_1^{\mathrm{st}} \oplus E_0^{f_1})) = w_2(2(E_1 \oplus \varepsilon^2)) + w_2(2(E_{-1} \oplus E_{-1})) = 0$. $\square$

Theorem 5.5. *For every closed connected surface Σ, there is an isomorphism of non-unital rings*

$$\widetilde{KO}_{\mathrm{com}}(\Sigma) \cong \widetilde{KO}(\Sigma) \times \langle y \rangle$$

where $2y = y^2 = 0$.

Proof. In the Appendix, we show that for non-simply connected surfaces, $\widetilde{KO}(\Sigma)$ is generated (as a ring) by stable line bundles. The classifying map $\Sigma \to BO(1)$ of a line bundle L is itself an algebraic TC structure on L (since $O(1)$ is abelian), and we refer to this as the *standard* TC structure. Whitney sum and tensor product of line bundles have preferred algebraic TC structures as well, since in this case block sum of matrices and Kronecker product are homomorphisms of abelian groups; we again refer to these preferred structures as *standard*. Associating to each generator $l_1, \ldots, l_k$ of the presentation of $\widetilde{KO}(\Sigma)$ given in Appendix B its standard (algebraic) TC structure yields a commutative diagram of ring homomorphisms

$$\begin{tikzcd} \mathbb{Z}[l_1, \ldots, l_k] \arrow[r, "s"] \arrow[dr, "\pi"'] & KO_{\mathrm{com}}(\Sigma) \arrow[d, "\iota_*"] \\ & KO(\Sigma) \end{tikzcd}$$

By commutativity, the kernel of π maps to the kernel of ι_* (note here that this kernel is the same whether we work with reduced or unreduced K–theory, and hence we use the notation ι_* in both settings). Each class in the image of s is algebraic, and by Remark 5.3 we know that a_2 vanishes on all such classes. By Theorem 4.9 there is only one non-zero class in $\ker(\iota_*)$, and a_2 is non-zero on this class. It follows that s must be zero on the kernel of π, and hence s factors through $\mathbb{Z}[l_1,\ldots,l_k]/\ker(\pi)\cong KO(\Sigma)$. We thus have induced maps

$$\sigma\colon\ KO(\Sigma)\longrightarrow KO_{\mathrm{com}}(\Sigma)$$

and

$$\widetilde{\sigma}\colon\ \widetilde{KO}(\Sigma)\longrightarrow \widetilde{KO}_{\mathrm{com}}(\Sigma)$$

such that the composition

$$KO(\Sigma)\xrightarrow{\sigma} KO_{\mathrm{com}}(\Sigma)\xrightarrow{\iota_*} KO(\Sigma)$$

is the identity, and similarly for $\widetilde{\sigma}$. This implies that we have a splitting of abelian groups

$$\widetilde{KO}_{\mathrm{com}}(\Sigma)\cong \widetilde{KO}(\Sigma)\oplus\ker(\iota_*),$$

and also shows that summand of $\widetilde{KO}_{\mathrm{com}}(\Sigma)$ corresponding to $\widetilde{KO}(\Sigma)$ (that is, the image of $\widetilde{\sigma}$) is a subring of $\widetilde{KO}_{\mathrm{com}}(\Sigma)$. Since $\ker(\iota_*)$ is isomorphic to $\mathbb{Z}/2$ as an abelian group, to complete the proof it suffices to show that for every element $x\in\widetilde{KO}_{\mathrm{com}}(\Sigma)$, the product of x with the generator $y\in\ker(\iota_*)$ is zero.

The case $\Sigma=S^2$ follows from Lemma 5.4. For the non-simply connected case, by Theorem 4.9 it is enough to show that

$$(c^*(E_0^{f_1})-(\varepsilon^2)^{\mathrm{st}})^2=0 \tag{11}$$

and that for an arbitrary stable line bundle L,

$$(c^*(E_0^{f_1})-(\varepsilon^2)^{\mathrm{st}})(L^{\mathrm{st}}-(\varepsilon^1)^{\mathrm{st}})=0. \tag{12}$$

Equation (11) follows from Lemma 5.4, since the collapse map $\Sigma\xrightarrow{c}S^2$ induces a ring homomorphism $c^*\colon\widetilde{KO}_{\mathrm{com}}(S^2)\to\widetilde{KO}_{\mathrm{com}}(\Sigma)$. To prove (12), it suffices to show that $(c^*E_0^{f_1}\otimes L^{\mathrm{st}})$ and $(2L^{\mathrm{st}}\oplus c^*E_0^{f_1})$ are stably equivalent. Both underlying bundles are stably equivalent to $2L$, and since there is a single non-standard class in $\widetilde{KO}_{\mathrm{com}}(\Sigma)$ that is detected by a_2, we only have to calculate the values of a_2 on these bundles. Using formula (9), we have

$$\begin{aligned} a_2(c^*E_0^{f_1}\otimes L^{\mathrm{st}}) &= w_2(2L)+w_2(c^*E_{-1}\otimes L)\\ &= w_2(2L)+c^*w_2(E_{-1})+w_1(L)^2\\ &= c^*(w_2(E_{-1})). \end{aligned}$$

One can readily verify that $a_2(2L^{\mathrm{st}}\oplus c^*E_0^{f_1})=c^*(w_2(E_{-1}))$ as well. □

Appendix A. The non-standard part of commutative K–theory

The inclusion $\iota\colon B_{\mathrm{com}}O\to BO$ induces a map $\iota_*\colon\widetilde{KO}_{\mathrm{com}}(X)\to\widetilde{KO}(X)$ for each finite CW complex X. The kernel of this map is the "non-standard" part of the commutative K–theory of X. In [4], it is shown that there is a splitting of infinite loop spaces

$$B_{\mathrm{com}}O\simeq BO\times E_{\mathrm{com}}O,$$

and this splitting yields an identification of $\ker(\iota_*)$ with $[X,E_{\mathrm{com}}O]$. The goal of this appendix is to give a more direct proof of this fact, using the Group Completion Theorem. While we focus on the orthogonal case, the same argument applies in the complex case, showing that $[X,E_{\mathrm{com}}U]$ represents the non-standard part of commutative complex K–theory.

Proposition A.1. *Let X be a finite CW complex. Then natural map*

$$[X, E_{\mathrm{com}}O] \to \ker(\iota_*)$$

is an isomorphism.

For the proof, we will need two lemmas.

Lemma A.2. *The spaces $E_{\mathrm{com}}O$, $B_{\mathrm{com}}O$, and BO are homotopy equivalent to CW complexes.*

Proof. The spaces $B_{\mathrm{com}}O$ and BO are colimits of the spaces $B_{\mathrm{com}}O(n)$ and $BO(n)$, respectively. The spaces $B_{\mathrm{com}}O(n)$ and $BO(n)$ are geometric realizations of good simplicial spaces that are triangulable at each level, and hence $B_{\mathrm{com}}O(n)$ and $BO(n)$ have the homotopy types of CW complexes by May [12, Corollary A.6]. The natural maps $B_{\mathrm{com}}O(n) \to B_{\mathrm{com}}O(n+1)$ and $BO(n) \to BO(n+1)$ are cofibrations (in general, level-wise cofibrations between good simplicial spaces realize to cofibrations; see Angelini-Knoll–Salch [1, Section 1], for instance), so the colimits $B_{\mathrm{com}}O$ and BO are homotopy equivalent to the corresponding mapping telescopes, and hence have the homotopy types of CW complexes by Milnor [17, Appendix].

Now consider $E_{\mathrm{com}}O$. In general, if $p\colon E \to B$ is a fibration between spaces that are homotopy equivalent to CW complexes, then the fibers $p^{-1}(b)$ ($b \in B$) are also homotopy equivalent to CW complexes (Schön [19, Proposition 3]). This also holds for the homotopy fibers of arbitrary maps $f\colon X \to Y$ between spaces of the homotopy types of CW complexes: one may convert f to a fibration $P_f \to Y$ whose fibers are the homotopy fibers of f, and the path space P_f has the homotopy type of a CW complex by Milnor [16, Theorem 3]. Since $E_{\mathrm{com}}O = \mathrm{hofib}(B_{\mathrm{com}}O \to BO)$, this completes the proof. □

Lemma A.3. *The spaces $E_{\mathrm{com}}O$, $B_{\mathrm{com}}O$, and BO are homotopy equivalent to identity components of loop spaces.*

In fact, these are *infinite* loop spaces by [4]; we give a direct proof of this simpler statement.

Proof. We will use some general facts about group completions of topological monoids; we refer to Ramras [18] for a more detailed discussion. Say M is a topological monoid such that $\pi_0 M$ is generated by the component of $m_0 \in M$. The McDuff–Segal approach to group completion [15] yields a diagram of the form

$$\Omega BM \xrightarrow{\simeq} \mathrm{hofib}\, q(M, m_0) \longleftarrow M_\infty(m_0), \tag{13}$$

where $M_\infty(m_0)$ is the infinite mapping telescope of the sequence

$$M \xrightarrow{-\cdot m_0} M \xrightarrow{-\cdot m_0} M \xrightarrow{-\cdot m_0} \cdots,$$

in which the maps are given by multiplication with m_0. The map $q(M, m_0)$ is built as follows. Whenever M acts on a space X, there is an associated translation category with object space X and morphism space $M \times X$, whose geometric realization we denote by X_M. When $X = *$, we have $X_M = BM$, and in general the projection $X \to *$ yields a natural map $X_M \to BM$. The left-multiplication action of M on itself extends to give an action of M on the mapping telescope $M_\infty(m_0)$, and $q(M, m_0)$ is the natural map $(M_\infty(m_0))_M \to BM$. The map $M_\infty(m_0) \to \mathrm{hofib}\, q(M, m_0)$ is simply the inclusion of the fiber over the base point into the homotopy fiber. If each path component of M has abelian fundamental group, then this map is a weak equivalence by Ramras [18, Proposition 4.4].

Now consider the monoid $M = \coprod_{n=0}^{\infty} B_{\mathrm{com}}O(n)$ or $M = \coprod_{n=0}^{\infty} BO(n)$, where the monoid operation is given by block sum of matrices (when $n = 0$, $B_{\mathrm{com}}O(n)$ and $BO(n)$ are defined to be one-point spaces, acting as formal identities under block sum). The spaces $B_{\mathrm{com}}O(n)$ and $BO(n)$ are path connected, and we have $\pi_1(B_{\mathrm{com}}O(n)) \cong \pi_1(BO(n)) \cong \mathbb{Z}/2$ by [4, Lemma 4.3]. Hence, taking m_0 to lie in $BO(1) = B_{\mathrm{com}}O(1)$, we see that M satisfies the above conditions, so that the right-hand map in (13) is a weak equivalence. We claim that for these monoids, all three spaces in (13) have the homotopy types of CW complexes. The mapping telescope $M_\infty(m_0)$ collapses down to the colimit $\mathbb{Z} \times B_{\mathrm{com}}O$ or $\mathbb{Z} \times BO$, and the collapse map is a homotopy equivalences (as discussed in the proof of Lemma A.2). The classifying spaces BM are good simplicial spaces (this holds whenever the inclusion of the identity element into M is a closed cofibration, and in the cases at hand, the identity element forms its own path component). Since M itself has the homotopy type of a CW complex in both cases, the same holds at each level of the simplicial space underlying BM. So May [12, Corollary A.6] implies that BM has the homotopy type of a CW complex, and Milnor [16, Theorem 3] shows that the same holds for ΩBM.

Next, we claim that for both of these monoids, the space $\mathrm{hofib}\, q(M, m_0)$ has the homotopy type of a CW complex. As in the proof of Lemma A.2, it suffices to show that $|T(M, M_\infty(m_0))|$ and BM have the homotopy types of CW complexes. Each level of the simplicial spaces underlying these spaces has the homotopy type of a CW complex, so it remains only to check that these simplicial spaces are good. But this again follows from the fact that the identity element in M is disjoint from the rest of M.

We have now shown that all of the spaces in the diagrams (13) are homotopy equivalent to CW complexes, so the maps in (13) are in fact homotopy equivalences (not just weak equivalences). Thus $B_{\mathrm{com}}O$ and BO are homotopy equivalent to $\Omega_0 B\left(\coprod_{n=0}^{\infty} B_{\mathrm{com}}O(n)\right)$ and $\Omega_0 B\left(\coprod_{n=0}^{\infty} B_{\mathrm{com}}O(n)\right)$, respectively, where $\Omega_0(-)$ denotes the identity component (that is, the path component of nullhomotopic loops).

Finally, we consider $E_{\mathrm{com}}O$. The inclusion $\coprod_{n=0}^{\infty} B_{\mathrm{com}}O(n) \hookrightarrow \coprod_{n=0}^{\infty} BO(n)$ now induces a commutative diagram connecting the diagrams (13) for these two monoids, and all four horizontal maps in this diagram are homotopy equivalences. It follows that the homotopy fiber $E_{\mathrm{com}}O = \mathrm{hofib}(B_{\mathrm{com}}O \xrightarrow{\iota} BO)$ is homotopy equivalent to the homotopy fiber of the map

$$\Omega_0 B\left(\coprod_{n=0}^{\infty} B_{\mathrm{com}}O(n)\right) \longrightarrow \Omega_0 B\left(\coprod_{n=0}^{\infty} BO(n)\right).$$

Since loop spaces and homotopy fibers are both pullbacks, we have

$$\Omega(\mathrm{hofib}(f\colon X \to Y)) \cong \mathrm{hofib}(\Omega f\colon \Omega X \to \Omega Y)$$

for any map $f\colon X \to Y$. Thus $E_{\mathrm{com}}O$ is homotopy equivalent to

$$\Omega_0\left(\mathrm{hofib}\left(\coprod_{n=0}^{\infty} B_{\mathrm{com}}O(n) \hookrightarrow \coprod_{n=0}^{\infty} BO(n)\right)\right).$$

□

Proof of Proposition A.1. We have a long exact sequence

$$[\Sigma X, B_{\mathrm{com}}O] \to [\Sigma X, BO] \to [X, E_{\mathrm{com}}O] \to [X, B_{\mathrm{com}}O] \to [X, BO],$$

where $[-,-]$ denotes the set of based homotopy classes of based maps (May [13, Section 8.6]). By Lemma A.3, the spaces $E_{\mathrm{com}}O$, $B_{\mathrm{com}}O$, and BO are (connected) H–spaces, so nothing changes if we use unbased maps instead. Moreover, Lemma A.3 tells us that each of these H–spaces admits a homotopy inverse, so the above sequence is in fact a sequence of *groups*. Exactness tells us that the map $[X, E_{\mathrm{com}}O] \to \ker(\iota_*)$ is surjective, and to complete the proof we must show that $[X, E_{\mathrm{com}}O] \to [X, B_{\mathrm{com}}O]$ is injective. As observed in [3], every vector bundle over a suspension admits a commutative trivialization. Hence the first map in the above sequence is surjective, and exactness implies that $[X, E_{\mathrm{com}}O] \to [X, B_{\mathrm{com}}O]$ is injective. □

Appendix B. Real topological K-theory of surfaces

In this section of the Appendix we describe the relationship between the $\mathbb{F}_2$-cohomology ring of a closed connected surface and its reduced real topological K-theory, yielding the following presentations:

$$KO(S^2) \cong \mathbb{Z}[e_1]/(2e_1, e_1^2);$$

$$KO(\Sigma_g) \cong \mathbb{Z}[l_{a_i}, l_{b_j} : 1 \le i, j \le g]/(2l_{a_i}, 2l_{b_j}, l_{a_i}l_{a_j}, l_{b_i}l_{b_j}, l_{a_i}l_{b_i}+l_{a_j}l_{b_j}, l_{a_i}l_{b_k} : i \ne k),$$

and

$$KO(P_n) \cong \mathbb{Z}[l_{a_i} : 1 \le i \le n]/(4l_{a_i}, l_{a_i}^2 - 2l_{a_j}, l_{a_i}l_{a_k} : i \ne k),$$

where Σ_g is the genus g oriented surface and P_n is the connected sum of n copies of $\mathbb{RP}^2$.

B.1. **Ring presentations.** Let R be a unital, commutative ring of characteristic zero that is additively generated by elements $r_1, \dots, r_n \in R$. Then there is a (unique) ring homomorphism $f\colon \mathbb{Z}[x_1, \dots, x_n] \to R$ sending x_i to r_i, and this homomorphism is surjective. For each pair $i, j \in \{1, \dots, n\}$, we have

$$r_i r_j = \sum_k a_k^{ij} r_k$$

for some $a_k^{ij} \in \mathbb{Z}$, and hence

$$x_i x_j - \sum_k a_k^{ij} x_k \in \ker(f).$$

Additionally, if $\sum_k a_k r_k = 0$ for some $a_1, \dots a_n \in \mathbb{Z}$, then

$$\sum_k a_k x_k \in \ker(f)$$

as well. A simple induction on degree shows that $\ker(f)$ is in fact generated by these elements, yielding a finite presentation of R. If R has characteristic $p > 0$, then we obtain a similar presentation by replacing $\mathbb{Z}$ with $\mathbb{F}_p$.

More generally, say $r_1, \dots, r_n$ generate R as a ring (but not necessarily as an abelian group). If

$$r_1, \dots, r_n, p_1(r_1, \dots, r_n), \dots, p_k(r_1, \dots, r_n)$$

form an additive generating set (where the p_i are integer polynomials), then the kernel of the surjection

$$f\colon \mathbb{Z}[x_1,\dots,x_n] \to R$$

sending x_i to r_i is the image of the kernel of

$$g\colon \mathbb{Z}[x_1,\dots,x_n,p_1,\dots,p_k] \to R$$

under the map sending p_i to $p_i(x_1,\dots,x_n)$ (since this map is a left inverse to the inclusion). Thus the generating set for $\ker(f)$ given above provides us with a generating set for $\ker(g)$. This again yields a finite presentation of R, which will be used to obtain the presentations in this Appendix.

B.2. **The total Stiefel–Whitney class and $\widetilde{KO}(-)$ for surfaces.** For a closed connected surface Σ, we let $x_2 \in H^2(\Sigma;\mathbb{F}_2) \cong \mathbb{F}_2$ denote the generator. The classes in the ungraded cohomology ring $H^*(\Sigma;\mathbb{F}_2)$ can be uniquely written as sums $x_0+x_1+x_2$, where $x_0\in\mathbb{F}_2$ and $x_1\in H^1(\Sigma;\mathbb{F}_2)$. The multiplicative units in the cohomology ring are then of the form $1+x_1+x_2$, and form an $H^*(\Sigma;\mathbb{F}_2)^\times$ abelian group under the cup product. The total Stiefel–Whitney class W of real vector bundles over Σ takes values in $H^*(\Sigma;\mathbb{F}_2)^\times$, and extends to a well–defined map

$$W\colon KO(\Sigma)\to H^*(\Sigma;\mathbb{F}_2)^\times$$

given by $W(E-F)=W(E)W(F)^{-1}$. Moreover, since $H^*(\Sigma;\mathbb{F}_2)$ is a commutative ring, W is a homomorphism. Restricting W to $\widetilde{KO}(\Sigma)$ and expressing a stable rank n bundle as $E-\varepsilon^n\in\widetilde{KO}(\Sigma)$ where $n=\mathrm{rank}(E)$ and ε^n is the trivial bundle of rank n, we see that $W(E-\varepsilon^n)=W(E)$.

Lemma B.1. *Let Σ be a closed connected surface. Then W induces an isomorphism of abelian groups*

$$W\colon \widetilde{KO}(\Sigma)\xrightarrow{\cong} H^*(\Sigma;\mathbb{F}_2)^\times.$$

Proof. First we show that W is surjective. For every $x_1\in H^1(\Sigma;\mathbb{F}_2)$, there is a line bundle L such that $w_1(L)=x_1$ and thus $W(L)=1+x_1$. The (stable) non-trivial rank 2 bundle c^*E_1 classified by the map $\Sigma\xrightarrow{c}S^2\xrightarrow{g_1}BSO(2)$ satisfies $w_2(c^*E_1)=x_2$ and $w_1(c^*E_1)=0$, so that $W(c^*E_1)=1+x_2$. For any unit of the form $1+x_1+x_2$ we now see that

$$W(c^*E_1\oplus L)=W(c^*E_1)W(L)=(1+x_2)(1+x_1)=1+x_1+x_2$$

and thus W is onto.

To see that W is injective, consider the cofiber sequence $\bigvee_n S^1\to\Sigma\xrightarrow{c}S^2$. The induced exact sequence in K–theory has the form

$$\mathbb{Z}/2=\widetilde{KO}(S^2)\to\widetilde{KO}(\Sigma)\to\widetilde{KO}\left(\bigvee_n S^1\right)=(\mathbb{Z}/2)^n,$$

which implies that $\widetilde{KO}(\Sigma)$ has at most 2^{n+1} elements. Since $H^1(\Sigma;\mathbb{F}_2)$ has rank n and $H^2(\Sigma;\mathbb{F}_2)$ has rank 1, $H^*(\Sigma;\mathbb{F}_2)^\times$ has exactly 2^{n+1} elements. Since W is surjective, it must be injective as well. ∎

B.3. **The abelian group structure of $H^*(\Sigma;\mathbb{F}_2)^\times$.** There is a general procedure for computing the $\mathbb{F}_2$-cohomology ring of a connected sum of surfaces $M_1\#M_2$ by analyzing the surjection in cohomology $H^*(M_1\vee M_2;\mathbb{F}_2)\to H^*(M_1\#M_2;\mathbb{F}_2)$ via the associated Mayer–Vietoris sequence.

Let Σ_g denote an oriented closed and connected surface of genus g. The above method shows that the graded $\mathbb{F}_2$–cohomology ring of Σ_g has a presentation

$$H^*(\Sigma_g;\mathbb{F}_2)\cong\mathbb{F}_2[a_1,...,a_g,b_1,...,b_g]/(a_ia_j,b_ib_j,a_ib_i+a_jb_j,a_ib_k:i\neq k),$$

where $\deg(a_i)=\deg(b_i)=1$.

Lemma B.2. *For every $g\geq 1$, there is an isomorphism of abelian groups*

$$H^*(\Sigma_g;\mathbb{F}_2)^\times\cong(\mathbb{Z}/2)^{2g+1}.$$

Moreover, $\{1+a_i,1+b_j,1+x_2:1\leq i,j\leq 2g\}$ generates $H^*(\Sigma_g;\mathbb{F}_2)^\times$.

Proof. Since $H^*(\Sigma_g;\mathbb{F}_2)^\times$ has 2^{2g+1} elements, to see that it is isomorphic to the group $(\mathbb{Z}/2)^{2g+1}$ it is enough to show that every element has order 2. All classes a_i and b_j square to zero, and thus for an arbitrary element $x_1\in H^1(\Sigma_g;\mathbb{F}_2)$, we find that the product $(1+x_1)(1+x_1)=1+x_1^2=1$. Similarly, the generator $x_2\in H^2(\Sigma_g;\mathbb{F}_2)$ squares to zero as well, and thus we have

$$(1+x_1+x_2)(1+x_1+x_2)=1+x_1^2+x_2^2=1.$$

To prove the final statement of the lemma, first note that if $i_1,\dots,i_k$ are distinct and $j_1,\dots,j_l$ are distinct, then

$$(1+a_{i_1})\cdots(1+a_{i_k})(1+b_{j_1})\cdots(1+b_{j_l})=1+a_{i_1}+\cdots+a_{i_k}+b_{j_1}+\cdots+b_{j_l}+\epsilon x_2,$$

where ϵ is zero if $|\{i_1,\dots,i_k\}\cap\{j_1,\dots,j_l\}|$ is even, and ϵ is 1 otherwise. Now

$$(1+a_{i_1})\cdots(1+a_{i_k})(1+b_{j_1})\cdots(1+b_{j_l})(1+\epsilon x_2)=1+a_{i_1}+\cdots+a_{i_k}+b_{j_1}+\cdots+b_{j_l}$$

and

$$\begin{aligned}(1+a_{i_1})\cdots(1+a_{i_k})&(1+b_{j_1})\cdots(1+b_{j_l})(1+(1-\epsilon)x_2)\\&=1+a_{i_1}+\cdots+a_{i_k}+b_{j_1}+\cdots+b_{j_l}+x_2,\end{aligned}$$

completing the proof. □

Let P_n denote the connected sum of n copies of $\mathbb{RP}^2$; note that P_n is non-orientable, and in fact each closed, connected, non-orientable surface is homeomorphic to P_n for some $n\geq 1$. The graded $\mathbb{F}_2$–cohomology ring of P_n has a presentation

$$H^*(P_n;\mathbb{F}_2)\cong\mathbb{F}_2[a_1,...,a_n]/(a_i^3,a_i^2+a_j^2,a_ia_k:i\neq k),$$

where $\deg(a_i)=1$. (Note that the relation $a_i^3=0$ follows from the other relations when $n>1$.)

Lemma B.3. *For every $n\geq 1$, there is an isomorphism of abelian groups*

$$H^*(P_n;\mathbb{F}_2)^\times\cong\mathbb{Z}/4\times(\mathbb{Z}/2)^{n-1}.$$

Moreover, $\{1+a_i:1\leq i\leq n\}$ generates $H^*(P_n;\mathbb{F}_2)^\times$.

Proof. We claim that the elements $1+a_{i_1}+\cdots+a_{i_k}$ where k is *odd* (and $i_1,\dots,i_k$ are distinct), form a collection of 2^{n-1} elements of order 4. This set has 2^{n-1}

elements, since for every non-empty finite set S, exactly half the elements have odd cardinality.[1] Now, for $i \geq 1$ we have $a_i^2 = x_2$, and then

$$(1 + a_{i_1} + \cdots + a_{i_k})^2 = 1 + x_2 + (a_{i_2} + \cdots a_{i_k})^2 = 1 + x_2,$$
$$(1 + x_2)(1 + a_{i_1} + \cdots + a_{i_k}) = 1 + a_{i_1} + \cdots + a_{i_k} + x_2,$$

and $(1 + x_2)^2 = 1$. Furthermore, when k is odd all elements of the form

$$1 + a_{i_1} + \cdots + a_{i_k} + x_2$$

also have multiplicative order 4, so we have $2(2^{n-1}) = 2^n$ elements of order 4 in $H^*(P_n; \mathbb{F}_2)^\times$. When k is *even*, the elements $1+a_{i_1}+\cdots+a_{i_k}$ and $1+a_{i_1}+\cdots+a_{i_k}+x_2$ (with $i_1, \ldots, i_k$ distinct) have order 2 (except in the first case, when $k = 0$ gives the identity element 1). This yields $2^n - 1$ elements of order 2 in $H^*(P_n; \mathbb{F}_2)^\times$. The only abelian group of order 2^{n+1} with $2^n - 1$ elements of order 2 and 2^n elements of order 4 is $\mathbb{Z}/4 \times (\mathbb{Z}/2)^{n-1}$.

The final statement in the lemma follows from the equation $(1 + a_i)^2 = (1 + x_2)$ (which holds for all i) together with the equation

$$(1 + a_{i_1} + \cdots + a_{i_k} + \epsilon x_2) = (1 + a_{i_1}) \cdots (1 + a_{i_k})(1 + \epsilon x_2),$$

which holds for both values of $\epsilon \in \mathbb{F}_2$ so long as $i_1, \ldots, i_k$ are distinct. □

B.4. **The ring structure of $KO(\Sigma)$.** We study the products in $\widetilde{KO}(\Sigma)$ using the total Stiefel–Whitney class. We view $\widetilde{KO}(\Sigma)$ as the kernel of $KO(\Sigma) \to KO(\mathrm{pt})$, so that elements in $\widetilde{KO}(\Sigma)$ are represented by stable bundles $E - \epsilon^n$, where $n = \dim(E)$.

Case $\Sigma = S^2$: The only non-trivial stable bundle over S^2 is $E_1 - \varepsilon^2$, and $W(E_1 - \varepsilon^2) = 1 + x_2$. Then

$$\begin{aligned} W((E_1 - \varepsilon^2)(E_1 - \varepsilon^2)) &= W(((E_1 \otimes E_1) \oplus \varepsilon^4) - (2(E_1 \oplus E_1))) \\ &= W(E_1 \otimes E_1) W(2(E_1 \oplus E_1))^{-1} \\ &= W(E_1 \otimes E_1)(1) \end{aligned}$$

The bundle E_1 has vanishing first Stiefel–Whitney class, then using formula (8) we conclude that $W(E_1 \otimes E_1) = 1$. By setting $e_1 = E_1 - \varepsilon^2$, we now see that $e_1^2 = 0$ in $\widetilde{KO}(S^2)$.

Case $\Sigma = \Sigma_g$: Let $L_{a_i} - \varepsilon^1$, $L_{b_j} - \varepsilon^1$ and $E_{x_2} - \varepsilon^2$ in $\widetilde{KO}(\Sigma_g)$ denote the inverse image under W of the units $1 + a_i$, $1 + b_j$ and $1 + x_2$, respectively. By Lemmas B.1 and B.2, these classes additively generate $KO(\Sigma_g)$, so the method from Section B.1 yields a presentation of $KO(\Sigma_g)$ with these elements as generators. We now compute the relations in this presentation.

First we notice that

$$\begin{aligned} W((L_{a_i} - \varepsilon^1)(L_{b_i} - \varepsilon^1)) &= W(((L_{a_i} \otimes L_{b_i}) \oplus \varepsilon^1) - (L_{a_i} \oplus L_{b_i})) \\ &= W(L_{a_i} \otimes L_{b_i})(W(L_{a_i} \oplus L_{b_i}))^{-1} \\ &= 1 + w_1(L_{a_i}) w_1(L_{b_i}) \\ &= 1 + x_2 \end{aligned}$$

[1] Symmetric difference with a one-element subset $\{s\} \subset S$ yields a bijection between the subsets of odd and even cardinality.

where $(W(L_{a_i} \oplus L_{b_i}))^{-1} = W(L_{a_i} \oplus L_{b_i}) = 1 + w_1(L_{a_i}) + w_1(L_{b_i}) + w_1(L_{a_i})w_1(L_{b_i})$. Hence we have the relation $(L_{a_i} - \varepsilon^1)(L_{b_i} - \varepsilon^1) = (E_{x_2} - \varepsilon^2)$. Similarly, for any i, j and k, with $i \neq k$ we see that

$$W((L_{a_i} - \varepsilon^1)(L_{a_j} - \varepsilon^1)) = 1 + w_1(L_{a_i})w_1(L_{a_j}) = 1;$$
$$W((L_{b_i} - \varepsilon^1)(L_{b_j} - \varepsilon^1)) = 1 + w_1(L_{b_i})w_1(L_{b_j}) = 1, \text{ and}$$
$$W((L_{a_i} - \varepsilon^1)(L_{b_k} - \varepsilon^1)) = 1 + w_1(L_{a_i})w_1(L_{b_k}) = 1.$$

For each $i, j \leq 2g$, set $l_{a_i} := L_{a_i} - \varepsilon^1$, $l_{b_j} := L_{b_j} - \varepsilon^1$, and $e_2 := E_{x_2} - \varepsilon^2$. The presentation arising from the method in Section B.1 is readily reduced to the simpler presentation

$$KO(\Sigma_g) = \mathbb{Z}[l_{a_i}, l_{b_j}]/(2l_{a_i}, 2l_{b_j}, l_{a_i}l_{a_j}, l_{b_i}l_{b_j}, l_{a_i}l_{b_i} + l_{a_j}l_{b_j}, l_{a_i}l_{b_k} : i \neq k).$$

Case $\Sigma = P_n$: For $1 \leq i \leq n$ let $L_{a_i} - \varepsilon^1 \in \widetilde{KO}(P_n)$ denote the inverse image under W of the unit $1 + a_i$. As in the previous case, these elements form an additive generating set for $KO(P_n)$. We have

$$\begin{aligned} W((L_{a_i} - \varepsilon^1)(L_{a_i} - \varepsilon^1)) &= W(L_{a_i} \otimes L_{a_i})(W(L_{a_i} \oplus L_{a_i}))^{-1} \\ &= (1)(1 + w_1(L_{a_i})^2) \\ &= (1 + x_2) \\ &= (1 + w_1(L_{a_1}))^2 \\ &= W(2(L_{a_1} - \varepsilon^1)), \end{aligned}$$

where we use that $W(L_{a_i} \oplus L_{a_i}) = 1 + w_1(L_{a_i})^2$ has (multiplicative) order 2. For $i \neq k$ we have that

$$W((L_{a_i} - \varepsilon^1)(L_{a_k} - \varepsilon^1)) = 1 + w_1(L_{a_i})w_1(L_{a_k}) = 1.$$

For $1 \leq i \leq n$, set $l_{a_i} := L_{a_i} - \varepsilon^1$. We then obtain the presentation

$$KO(P_n) = \mathbb{Z}[l_{a_i}]/(4l_{a_i}, l_{a_i}^2 - 2l_{a_j}, l_{a_i}l_{a_k} : i \neq k).$$

References

[1] G. Angelini-Knoll, A. Salch. “Maps of simplicial spectra whose realizations are cofibrations,” arXiv:arXiv:1611.06215, preprint 2018.

[2] A. Adem, F. Cohen, E. Torres-Giese. “Commuting elements, simplicial spaces and filtrations of classifying spaces”, Math. Proc. Cambridge Philos. Soc. Vol. 152 (2012), 1, 91-114.

[3] A. Adem, J. M. Gómez. “A classifying space for commutativity in Lie groups,” Algebraic & Geometric Topology 15 (2015) 493-535.

[4] A. Adem, J. M. Gómez, J. Lind, U. Tillman. “Infinite loop spaces and nilpotent K-theory,” Algebraic & Geometric Topology 17 (2017) 869-893.

[5] O. Antolín-Camarena, S. P. Gritschacher, B. Villarreal. “Classifying spaces for commutativity in low-dimensional Lie groups,” arXiv:1802.03632, preprint 2018.

[6] O. Antolín-Camarena, B. Villarreal. “Nilpotent n-tuples in $SU(2)$,” arXiv:1611.05937, preprint 2017.

[7] J. Ebert, O. Randall-Williams. “Semi-simplicial spaces”, arXiv:1705.03774, preprint 2017.

[8] J. M. Gómez, A. Pettet, J. Suoto. “On the fundamental group of Hom($\mathbb{Z}^k, G$),” Math. Z. (2012) 271:33-44.

[9] S. P. Gritschacher. “Commutative K-theory,” PhD thesis, University of Oxford, 2017.

[10] G. Higuera Rojo. “On the space of commuting orthogonal matrices,” J. Group Theory 17 (2014), 291-316.

[11] G. Segal. “Categories and cohomology theories,” Topology Vol. 13 (1974), pp. 293-312.

[12] J. P. May. "E_∞ spaces, group completions, and permutative categories." In *New developments in topology (Proc. Sympos. Algebraic Topology, Oxford, 1972)*, pp. 61-93. London Math. Soc. Lecture Note Ser., No. 11. Cambridge Univ. Press, London, 1974.

[13] May, J. P. *A concise course in algebraic topology.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1999. x+243 pp.

[14] J. McCleary. *A user's guide to spectral sequences.* Second edition. Cambridge Studies in Advanced Mathematics, 58. Cambridge University Press, Cambridge, 2001. xvi+561 pp.

[15] D. McDuff and G. Segal. Homology fibrations and the "group-completion" theorem. *Invent. Math.*, 31(3):279–284, 1975/76.

[16] Milnor, J. "On spaces having the homotopy type of a CW–complex," Trans. Amer. Math. Soc. 90 (1959) 272280.

[17] Milnor, J. *Morse theory.* Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51 Princeton University Press, Princeton, N.J. 1963 vi+153 pp.

[18] D. A. Ramras, "The homotopy groups of a homotopy group completion," preprint 2017, https://www.math.iupui.edu/~dramras/research.html.

[19] Schön, R. "Fibrations over a CWh–base," Proc. Amer. Math. Soc. 62 (1977), no. 1, 165-166.

[20] E. Torres Giese and D. Sjerve. "Fundamental groups of commuting elements in Lie groups," Bull. London Math. Soc. 40 (2008) 65-76.

Mathematical Sciences, Indiana University - Purdue University Indianapolis, IN 46202
E-mail address: dramras@iupui.edu

E-mail address: bervilla@iu.edu